Manuscript for a course in

General Equilibrium Policy Evaluation

Spring 2015

at the

Institute for Advanced Studies, Vienna

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Vienna, Spring 2015

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Contents

Chapter 1

Introduction

This script was written for a course on 'General Equilibrium Policy Evaluation'. The aim of the course is to introduce students to (a class of) **computable dynamic general equilibrium** models in order to address questions typically asked in the field of applied economics. In particular the focus is given to the quantitative assessment of government policy related to the tax, social security and pension system. Examples of the type of questions are: "*What are the medium-run effects on GDP of revenue-neutrally shifting emphasize from the income to the value added tax in Spain?*", "*How much would the effective retirement age in Austria have to change over time to offset the effects from increasing population aging keeping all other pension system parameters constant?*" or "*What are the consequences of increasing yearly migration flows to the United Kingdom to its public budgets?*". This task requires a rather complete modeling approach with a realistic incorporation of an economy's institutional setting, etc. The required complexity typically goes far beyond to what can still be solved by just using pencil and paper. The course introduces students to numerical solution techniques and how to convert economic problems such that they can be implemented on the computer. Those techniques are introduced from scratch. That means that we do cover the application of higher modeling and solution tool boxes such as GAMS. The only requirements are the most basic numerical operation as provided by many numerical computing environments. All codes are provided for MATLAB[1] but do not rely on any MATLAB-specific func-

[1]For a quick introduction to MATLAB check out the 'MATLAB Primer' available at `http://www.mathworks.com`.

tions such that the codes could easily be translated to many other languages.

Scope. Dynamic general equilibrium (DGE) models are the workhouses of modern macroeconomics and their applications range over many different fields. The course is not intended to give a general overview of all possible applications. Instead we focus on a particular subset of model specifications. The covered model characteristics are: discrete timing, agents with rational expectations and perfect foresight that optimize inter-temporally, general equilibrium, households of overlapping generations and a focus on fiscal policy. The following features are not covered. The course does not address aggregate shocks and dynamic *stochastic* general equilibrium (DSGE) models especially used to assess the role of short-run business cycle fluctuations. True intra-cohort heterogeneity, e.g. stemming from idiosyncratic shocks to employment, income, skills, etc., which leads to a distribution of life-cycle profiles[2] and much more complicated state spaces are not covered in the course. Hence, distributional questions will be addressed based on inter- and not on intra-generational heterogeneity. Further, all discussed models are 'real', i.e. nominal effects and monetary policy do not play a role. The presentation is further restricted to a positive analysis, i.e. we do not attempt to answer normative questions. The reduction of complexity in the described areas obviously leaves room for a more complete modeling of and stronger focus on other areas, e.g. a detailed modeling of the public sector and a reasonably fast computation of transitional dynamics. Heer and Maussner (2009) provide a thorough discussion of DGE models that goes far beyond the cases considered in this course and is highly recommended for interested readers.

How to use this script. The script is meant to be used as a form of technical appendix. The main purpose is to document most of the algebraic steps that have to be made before implementation on the computer can start. In class we will rather focus on the implementation process, issues of simulation,

[2] A common approximation is to allow for idiosyncratic shocks, resulting e.g. in a distribution of household incomes during a period, but to assume that for the intertemporal optimization the households behave like a representative one by assuming that they pool income for the savings decision. Andolfatto (1996) is an example. In contrast, the strand of the literature labeled as *heterogenous agent modeling* does not rely on this simplifying assumption.

debugging, calibration, etc. and on the interpretation of the simulation results. The lecture starts with solving a very simple model. Once the solution technique is mastered we will step by step introduce additional model components. All additional components are thoroughly described in the script. In addition it contains exercises to be done after class.

Computer Code. Codes for all the discussed models and simulations are available in MATLAB. Material can be downloaded from `https://sites.google.com/site/schusterphilip/`.

Acknowledgments. The content of this course is based on my work I did at the Institute for Advanced Studies. The institute runs a CGE model called TAXLAB (*tax* and *lab*or model) which is heavily used for policy advice and continuously refined. Although the complexity of TAXLAB goes far beyond the content covered in this course it shares the same core framework and numerical solution mechanism as the models presented in this script. The original development of this framework was done by Christian Keuschnigg and the lecture notes are based on his set-up work and a series of technical notes provided by him. Please address all comments and feedback to the notes to `philip.schuster@oenb.at`.

1.1 Overview and Background

The outline of the script is as follows. Chapter 2 introduces the reader to the Fair-Taylor algorithm which is used to solve for transition paths between two steady states. It is based on characterizing the solution to a dynamic model as a system of first-order difference equations, i.e. the solution is expressed in recursive form just like the problem itself is often conveniently expressed recursively (see Stokey and Lucas, 1989 or Ljungqvist and Sargent, 2012). While some variables just summarize past decisions, like a capital stock or a stock of pension entitlements, there are other variables reflecting future decisions such as future profits that determine today's decisions. In a nutshell the algorithm solves the model forward by using naive guesses about the foresight variables (usually based on the final steady state). Afterwards we check by how much the realized foresight variables differ from the guesses and use this information to make a more informed guess about the future in the next iteration and so on until convergence, i.e. until we have found the time-consistent values of our foresight variables.

In chapter 3 we apply this technique to solve for the transition paths of a simple Ramsey model (see Ramsey, 1928 or Romer, 2011) to illustrate the procedure. The way this algorithm works will be always the same irrespective of the complexity of the models. The computations might involve more markets for which we have to find the clearing prices or many more foresight variables whose paths are ex-ante unknown but the main structure of our approach will be unchanged. After this introduction we will leave the realm of working with a single, infinitely-lived household. Based on an overlapping generations household structure we will step by step add additional model components. Chapter 4.1 introduces mortality which households face with a constant probability, the so-called Blanchard (1985)-model. This is the only true idiosyncratic shock that is incorporated in the presented models, as it is rather trivial to handle.[3] Given certain assumptions this model allows for analytical aggregation which from a numerical perspective makes it virtually as easy to solve as the Ramsey model. The chapter continues to present the difference of a closed and a small open economy setting and how changes in the deep parameters of the demographic process translate into the change in macroeconomic variables. The next section introduces endogenous labor supply along the intensive and the extensive margin which is necessary for the following introduction of government policy. Various tax instruments are incorporated at every decision margin which allows first simple fiscal policy evaluation exercises and simulations. Chapter 5 presents a generalization of the Blanchard model by differentiating between two age classes (workers and retirees) which allows to model the existence of a pension system, while still keeping the model quite tractable. The so-called Gertler (1999)-model is then even further generalized – namely to A age classes – in the 'Probabilistic Aging'-model by Grafenhofer et al. (2007), which is presented at the end of the chapter. With chapter 6 we enter the realm of large-scale numeri-

[3] In contrast to e.g. idiosyncratic income shocks we do not have to individually keep track of all past realizations of a random variable.

cal overlapping generations models by presenting the so-called Auerbach and Kotlikoff (1987)-model. The model is characterized by age groups of length of one year which will result in a much more detailed representation of the household sector. The Auerbach-Kotlikoff-model generates a more realistic age structure of the population and features more realistic income and consumption life-cycle profiles. The process of finding a reasonable calibration will be discussed before we will look at the inter-generational effects of fiscal policy and reforms of the pension system. The chapter will then close with several possible extensions without doing a full derivation of the solution. This section closes with a short revision of the model assumptions concerning the demography of different overlapping generations (OLG) models.

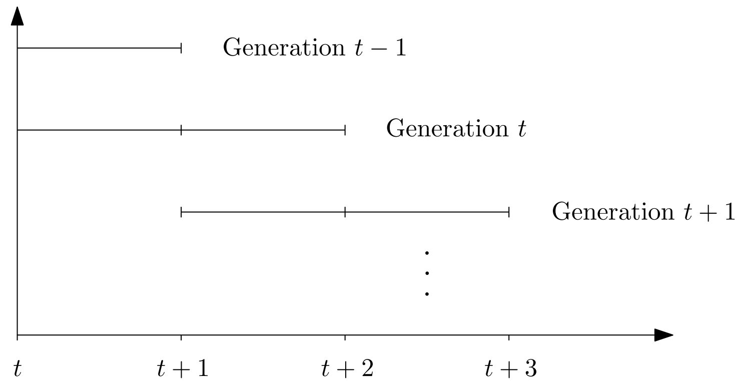
Overlapping Generations Models. A realistic age structure is an important feature of a model which is designed to answer public policy questions related to the household sector. The first formulation is due to Samuelson (1958) and Diamond (1965) where the population merely consists of two generations at every point in time, a young and an old cohort. A time period therefore has to comprise many years (e.g. 30 years). All households have the same (deterministic) life expectancy of twice the period's length. Denote N^O the mass of old and N^Y the mass of young households. The demographic transition rules are described as

$$N_{t+1}^O = N_t^Y, \tag{1.1.1}$$

$$N_{t+1}^Y = NB_{t+1}, \tag{1.1.2}$$

where NB are the number of newborns. Clearly, $NB_{t+1} = N_{t+1}^O$ implies that the population size is constant. Figure 1.1.1 illustrates the evolution of generations in the Diamond-Samuelson model. The original model by Auerbach and Kotlikoff (1987) is a natural extension of the Diamond-Samuelson approach by increasing the number of age groups. In contrast to the simple 2-period model the handling of the complexity was made possible only through the technological progress in computing power. Because of its finer representation of the household sector this model is much more appropriate to deliver quantitative results. The length of a period is usually set to one year. Setting the maximal attainable age to A the evolution of the age-specific cohort sizes

Figure 1.1.1: Cohort life spans over time in a Diamond-Samuelson model



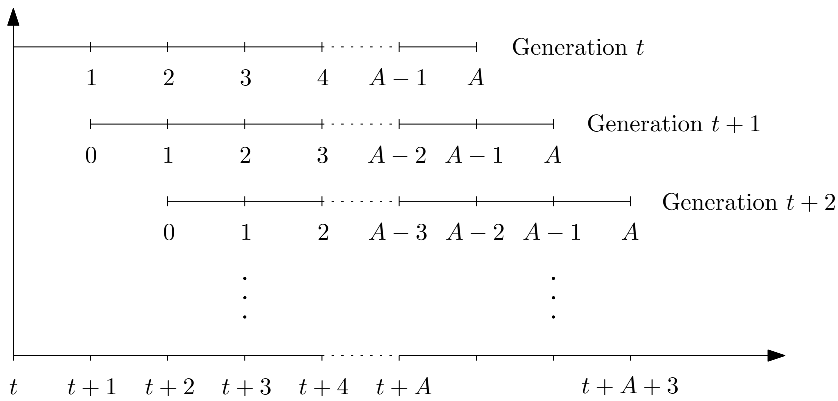
is given by

$$N_{t+1}^{a+1} = N_t^a,\ 0 \leq a < A \tag{1.1.3}$$

$$N_{t+1}^0 = NB_{t+1}. \tag{1.1.4}$$

As life expectancy at birth is equal to the maximum age A the age distribution is degenerate. Figure 1.1.2 presents this visually. The Blanchard

Figure 1.1.2: Cohort life spans over time in an Auerbach-Kotlikoff model



(1985)-model addresses overlapping generations differently. Instead of fixing a maximal attainable age households face a constant probability of death $(1-\gamma)$. Hence, the age distribution is non-degenerate and life expectancy can be computed as follows. The probability of dying at age 0 is $1-\gamma$, of dying at age 1 is $\gamma(1-\gamma)$, of dying at age 2 is $\gamma^2(1-\gamma)$, and so on. Hence, average age of death, i.e. life expectancy at birth, is given as

$$\text{life expectancy } = (1-\gamma)\sum_{a=0}^{\infty} a\gamma^a = (1-\gamma)\frac{\gamma}{(1-\gamma)^2} = 1/(1-\gamma) - 1.$$

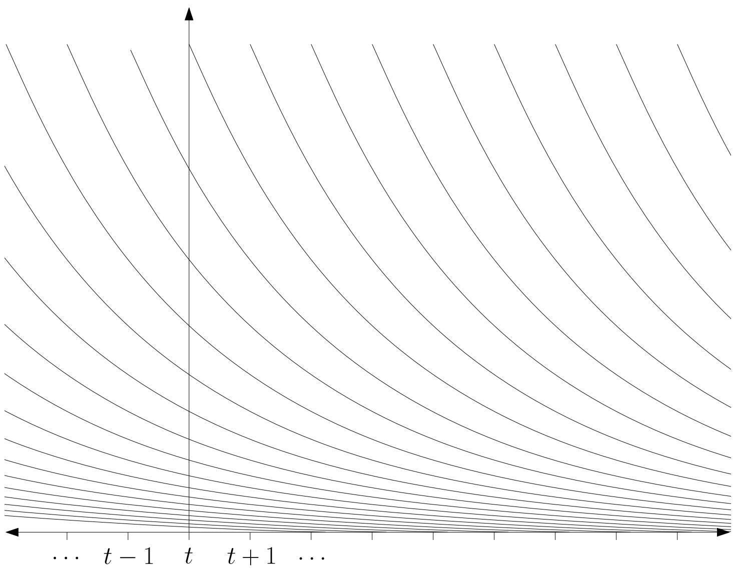
Note that in the Blanchard model the remaining life expectancy, i.e. the expected years remaining at a certain age, is always the same independent of the current age. Let $N_{v,t}$ denote the mass of households born at $v \leq t$ at time t. The evolution of cohorts is described by

$$N_{v,t+1} = \gamma N_{v,t}, \ \forall v \leq t \tag{1.1.5}$$

$$N_{t+1,t+1} = NB_{t+1}. \tag{1.1.6}$$

Every period a constant fraction of a generation dies such that it decays smoothly over time. At every period in time there are infinitely many generations and the mass of old generations converges to zero. This is illustrated in 1.1.3 where we plot the size of a generation over time. In steady state

Figure 1.1.3: Cohort sizes over time in a Blanchard model



deaths have to equal births which therefore gives us a relationship for total population size and the number of newborns, $N(1-\gamma) = NB$. One can then compute average age of the population in steady state. There are NB households with age 0, $NB \cdot \gamma$ with age 1, $NB \cdot \gamma^2$ with age 2, and so on.

Average age is therefore given by

$$\text{average age } = \frac{NB \cdot \sum_{a=0}^{\infty} a\gamma^a}{N} = \frac{NB \cdot \gamma/(1-\gamma)^2}{NB/(1-\gamma)} = 1/(1-\gamma) - 1.$$

Hence, in the Blanchard model average age in the steady state population is equal to the life expectancy at birth. This result will no longer hold for more realistic demographic models where survival rates are decreasing with age. In contrast to a purely deterministic demography the Blanchard (1985) assumption generates a non-degenerate age distribution. Households want to insure themselves against the risk of longevity. On the other hand working with infinitely many generations is difficult. However, we can obtain analytical results for the aggregate of households (under certain symmetry assumptions). Further, demographic shocks in the Blanchard model typically fade out at an unrealistically slow pace which is not the case in the Auerbach-Kotlikoff model. However, in its original specification the Auerbach-Kotlikoff model is purely deterministic as households can perfectly predict when they are going to die. In chapter 6 we will therefore focus on a mixture of both approaches by introducing mortality risk to the Auerbach-Kotlikoff model (see Broer and Lassila, 1997). In this case the evolution of cohorts is then described by

$$N_{t+1}^{a+1} = \gamma^a N_t^a, \; 0 \leq a < A, \text{ with } \gamma^A = 0, \tag{1.1.7}$$

$$N_{t+1}^{0} = NB_{t+1}. \tag{1.1.8}$$

Life expectancy is now given as $\sum_{a=1}^{A} a(1-\gamma^a)\prod_{s=0}^{a-1}\gamma^s$. Convince yourself that this results in the same life expectancy as stated above for the nested special case of a Blanchard model, i.e. $\gamma^a = \gamma, \; \forall a$ and $A \to \infty$.

$$\begin{aligned}\text{life expectancy} &= \sum_{a=1}^{\infty} a(1-\gamma)\,(\gamma)^a = (1-\gamma)\sum_{a=0}^{\infty} a\,(\gamma)^a \\ &= (1-\gamma)\frac{\gamma}{(1-\gamma)^2} = \frac{\gamma}{(1-\gamma)} = 1/(1-\gamma) - 1.\end{aligned}$$

In contrast to the Blanchard model average age in steady state will no longer coincide with life expectancy. The former is computed as

$$\text{average age} = \frac{\sum_{a=1}^{A} a \prod_{s=0}^{a-1} \gamma^s}{1 + \sum_{a=1}^{A} \prod_{s=0}^{a-1} \gamma^s}.$$

In principle an even further generalization is the idea of probabilistic aging by Grafenhofer et al. (2007). An important characteristic of this approach is that the duration a household stays in an age class is disconnected from length of a model period. This way one build a yearly model with realistic average ages which features only a few age classes. Let a be the index of an age-class with $a \in \{1, 2, ..., A\}$ and $1 - \omega^a$ the yearly probability of 'aging' into the next age group. The evolution of the age-classes is then described by

$$N_{t+1}^{a+1} = \gamma^{a+1}\omega^{a+1}N_t^{a+1} + \gamma^a(1-\omega^a)N_t^a, \ 1 \leq a < A, \ \text{with } \omega^A = 1, \quad (1.1.9)$$
$$N_{t+1}^1 = \gamma^1\omega^1 N_t^1 + NB_{t+1}. \quad (1.1.10)$$

The restriction $\omega^A = 1$ simply implies that a person in age-class A can no longer jump into a higher age-class and will stay there until he dies, i.e. the last age-class works as in the Blanchard model. Observe that this specification is the most flexible one as it can nest all other presented cases. The Blanchard model is nested through the specification $A = 1$. The Auerbach-Kotlikoff model with mortality is the special case where A is set to a large number (maximum age in years), all $\omega^a = 1$ and the last group dies with certainty, i.e. $\gamma^A = 0$. The Gertler (1999)-model is replicated by setting $A = 2$ and $\gamma^1 = 1$.

The demographic process can be further refined by relating the number of newborns to older generations through fertility. Papers have endogenized demographic parameters, e.g. mortality rates that depend on health investments or fertility that stems from an explicit optimization of households. Those formulations break the recursion of the model, i.e. demography cannot be separately computed anymore before solving the economic part of the model. This is not addressed in further detail in the course. Many possible OLG extensions and trends are discussed in Fehr (2009) who provides a recent survey. An example of an extremely heavy version of the Auerbach-

Kotlikoff model is Fehr et al. (2008). Next to a more realistic notion of a household (e.g. a child is born into a family and part of the household until it leaves and forms an own household, etc.) they consider many extensions such as different skill classes, six different good sectors, all embedded in a five countries setting, etc.

The CGE model of the Institute for Advanced Studies, TAXLAB, is a single country model (although calibrated for 14 different countries). The population consists of eight age groups (using the concept of probabilistic aging) and three skill classes. All representative households are calibrated based on micro data sets (EU-SILC and LFS). It has a strong focus on the household sector's labor supply decisions by featuring education choice, participation, hours and training decisions as well as involuntary unemployment. It was designed to incorporate a high level of institutional detail, e.g. a past-earnings-related pension system that consists of a PAYG and a capital-funded pillar. The model has an extension in which population is separated by nationality to address effects of migration and another to capture informal work and the shadow economy. The development of the model is documented in a series of papers Keuschnigg and Keuschnigg (2004), Grafenhofer et al. (2007), Berger et al. (2009), Jaag et al. (2010), Keuschnigg et al. (2011), Keuschnigg et al. (2012a) and Keuschnigg et al. (2012b). Despite this complexity the basic algorithm for solving a transition path is virtually identical to what is covered in the next sections of this manuscript.

1.1.1 Exercises

Exercises

Ex. 1 — Probabilistic Aging - Life expectancy

Find an analytical expression for life expectancy of a household in the probabilistic aging framework.

Chapter 2

Solving Deterministic Dynamic Perfect Foresight Models

There are different aspects of solving a dynamic deterministic perfect foresight model. Calculating **steady states** is a rather trivial exercise especially in comparison to their stochastic counterpart versions. However, computing the full transition from one steady state to another is an involved task. A task that will typically require an iterative algorithm in order to be solved. Assume that the economy is in an initial steady state at $t = 0$. We want to compute the transition starting at $t = 1$ to $t = T$, where T is assumed to be sufficiently large such that the economy at $t = T$ is almost indistinguishably close to a final steady state. Hence, solving for the transition path involves solving for a path of prices for all T periods. Assume that there are n markets to clear. An obvious option is to stack all market clearing conditions of all periods and solve this system in a single go using a method for solving systems of non-linear equations, e.g. a multidimensional Newton-Raphson method. For small models this will typically converge fast. However the size of the system can easily grow very large if a more complex model is considered, which will reduce the robustness of such an approach considerably.[1] Instead it makes sense to split the problem in a series of subproblems - one for every period along the transition path - which is described in more detail

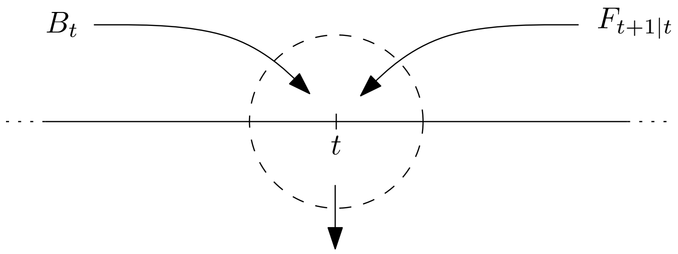
[1]First note that given the structure of the economic model it might be necessary to compute more roots than there are markets each period. To give the reader an idea, extended versions of the IHS TAXLAB model require finding the roots of close to 200 equations per period. For a transition time horizon of 200 years this would imply that one would have to solve a system of approximately 40'000 unknowns in one shot.

in the next section.

2.1 The Fair-Taylor Algorithm

Being able to split our problem into subproblems for every period requires that it can be expressed in a recursive form (see e.g. Stokey and Lucas, 1989 or Ljungqvist and Sargent, 2012). In discrete time most rational expectation models are basically just a system of first-order difference equations. Figure 2.1.1 illustrates the subproblem at time t in a general way. Importantly one has to distinguish between three type of variables: (a) predetermined, backward looking variables B (b) forward looking, foresight variables F and (c) exogenous parameters Z. A predetermined variable would for example be the capital stock. All investment decisions prior t are summarized in one variable at time t. An example for a foresight variable would be future wage income which can be an important determinant e.g. for today's consumption decision (depending on the assumptions concerning preferences). Foresight variables F therefore implicitly include expectation about future prices. A remark concerning notation. $F_{t+1|t}$ summarize future information (i.e. in expected terms) concerning foresight variables in $t+1$ available at t. An example would be the expectation at time t about all future labor income discounted to $t+1$. Z contains time paths for all exogenous variables, e.g. a path of tax rates. Given all predetermined variables and all guesses about the foresight

Figure 2.1.1: The subproblem at time t



p_t such that $\zeta_n(p_t, B_t, F_{t+1|t}, Z_t) = 0,\ \forall n$

$B_{t+1} = \Pi(p_t, B_t, F_{t+1|t}, Z_t)$

$F_{t|t} = \Gamma(p_t, B_t, F_{t+1|t}, Z_t)$

variables agents make optimal decisions, interact in markets leading to prices p_t that set all excess demands to zero, i.e. $\zeta_n(p_t, B_t, F_{t+1|t}, Z_t) = 0,\ \forall n$. We refer to this as being in **temporary equilibrium**. Knowing the behavior of agents in t and prices p_t one can compute the predetermined variables for $t+1$ using the abstract function $\Pi(\cdot)$ which represents the typical laws of motion, e.g. for capital or household assets. At the same time we can also compute the value of the foresight variables taking this period's decisions into account. For example if $F_{t+1|t}$ is the expectation at time t about all future labor from $t+1$ onward, then adding this period's labor income appropriately would give expected labor income from t onward, $F_{t|t}$. This last calculation is purely done in order to update information. The next paragraph explains this in more detail.

So far we have be silent about how expectations about future variables are formed. Assume that there was a naive expectation forming process, e.g. $F_{t+1|t}$ is simply determined by what happened in the past. Then the problem could easily be solved by starting at $t = 1$ solving the problem described in figure 2.1.1 and iterate forward until we reach $t = T$. However, this will generally imply that $F_{t|t} \neq F_{t|t-1}$, i.e. with a naive ex-ante guess about the future an agent would always want to revise her decision ex-post after observing the actual prices at t. In contrast with perfect foresight, price expectations are time consistent with the agents' behavior and the resulting actual prices. Hence, a transition path subject to rational expectations has to fulfill $F_{t|t} = F_{t|s\neq t}, \ \forall t$, i.e. expectations are self-fulfilling. In order to compute time consistent expectations proceed as follows. We start at $t = 1$ by making a guess for the whole path of foresight variables F^0, i.e. F^0 has dimension T times m, the number of different foresight variables. For a given F^0 we solve the subproblems described in figure 2.1.1 starting at $t = 1$ for the whole transition path. Along the way the guesses of the m foresight variables are updated at every t. After consecutively solving T subproblems we converted F^0 into an updated guess matrix F^1. Stepping through the subproblems from $t = 1$ to $t = T$ is denoted as function $\Lambda(\cdot)$, hence

$$F^{i+1} = \Lambda\left(F^i\right). \tag{2.1.1}$$

Repeating this will eventually lead to time consistent foresight variables $F^i = F^{i+1} \equiv F$. Note however that the existence of a fixed point of $\Lambda(\cdot)$ is in general not guaranteed. There can be economic reasons. For example, even though an initial and a final steady state can be computed Z_t could contain an extreme non-monotonic time path of certain parameters, e.g. tax rates that increase and later decrease again but in between reach a level that leads to break down of equilibrium. There can also be numerical reasons. For example, an uneducated guess of F^0 might imply failure of convergence. In order to improve stability of the algorithm it is generally implemented as

$$F^{i+1} = \psi \Lambda \left(F^i \right) + (1 - \psi) F^i. \tag{2.1.2}$$

The smaller $\psi \in (0, 1]$ the more stable the algorithm but also the slower. The algorithm described in (2.1.2) is know as the Fair and Taylor (1983) algorithm. Wilcoxen (1989) presented a generalized version of the algorithm which also uses information of the Jacobian of $\Lambda(\cdot)$ in order to improve the guess updating which reduces the number of i iterations for convergence.

2.2 Implementation on the Computer

The algorithm in order to solve a full problem including the transition path will always have the same structure independent of the complexity of the model itself.

1. load model parameters
2. calibrate the model
3. compute the Jacobian of the system $\Lambda(\cdot)$ (only necessary if generalized version of Fair-Taylor algorithm is used)
4. compute the initial steady state at $t = 1$
5. enter a shock to system, e.g. a policy reform
6. compute the final steady state at $t = T$
7. solve for the transition path
    a) make a guess for F

b) compute a path of temporary equilibria starting from $t = 1$ to $t = T - 1$

c) update the guess for F

d) repeat steps b) to c) until convergence

In the course all provided codes are done in MATLAB. For more complex model versions (which are not addressed in class) the use of languages more optimized for speed such as Fortran, C/C++, GAUSS, etc. is recommended. The principle structure of our programs will always look as shown in code section 1.

Code 1: The principle structure of our programs

```
% ----------------------
% runme.m
% ----------------------
% computes the transition path of a
% Ramsey model to a rise in labor supply
% ----------------------

clear all;  % clear all variables
clc;        % clear screen

% DATA AND CALIBRATION
global tend

tend    = 50; % number of periods

param; % loads exogenous parameters
calib; % calibrates the model

% COMPUTE JACOBIAN FOR DYNAMIC SOLUTIONS
jacob = compjacob(guess);

% COMPUTE INITIAL STEADY STATE
ISS;

% REFORMS
L0 = 1.1*L0;

```

```
% COMPUTE FINAL STEADY STATE
FSS;

% SOLVE FOR TRANSITION PATH
[actual,ret] = gft(guess,jacob);         % Generalized FT
%[actual,ret] = ft(guess);               % Original FT
```

The function `ft()` calls the Fair-Taylor algorithm and takes the following inputs and produces the following outputs

```
[actual,ret] = ft(guess,tune0,tstart,maxiter,tol)
```

`guess` ..... $T \times m$ matrix of the foresight variables
`tune0` ..... stability parameter ψ (default = 1)
`tstart` .... starting period of the algorithm (default = 1)
`maxiter`.... maximum number of iterations (default = 2000)
`tol` .... convergence criterion (default = 1e-06)
`actual` .... updated foresight variables after convergence
`ret` .... return code, 1 = convergence, 0 = no convergence

The function `gft()` calls the generalized Fair-Taylor algorithm and takes two additional input arguments

```
[actual,ret] = gft(guess,jacob,tune0,tune1,tstart,maxiter,tol)
```

`jacob` ..... $m \times m$ matrix of the Jacobian of $\Lambda(\cdot)$
`tune1` ..... stability parameter for the use of the Jacobian (default = 1)

The generalized Fair-Taylor algorithm delivers the same results as the regular Fair-Taylor method if `tune1 = 0`, in which case no information from the Jacobian of $\Lambda(\cdot)$ is used. Iteration over periods and updated guesses is split into two functions. A function called `path()` solves all the temporary equilibria `TE()` forward in time for a given matrix of guesses. `ft()` or `gft()` then iterate over `path()` by updating the guess matrix after every iteration. The content and structure of the functions `path()` and `TE()` are exemplarily discussed in the next section for a simple Ramsey model.

Chapter 3

A Simple Ramsey Model

3.1 Closed Economy

Summary

This section shows how to characterize the solution of a simple Ramsey model in a way that can be directly implemented and solved numerically on a computer. The presented model is of the following very reduced form. Labor supply is fixed. The representative household's felicity function is log. Production is handled by a single representative firm that invests using retained earnings. The firm is owned by the representative household. There is no government. The corresponding codes are `Ramsey` and `Ramsey_simple`.

3.1.1 Description of the Economy

The economy exists of a representative household that supplies labor (exogenously fixed at $L_t^S = L_0$) and owns all production facilities that produce a homogeneous good. There are three markets: goods, labor and assets, with prices 1, w and r.

3.1.2 Representative Household

The household maximizes life-time utility subject to the intertemporal budget constraint, i.e.

$$U_t \equiv \max_{\{C_s\}_t^\infty} \sum_{s=t}^{\infty} \beta^{s-t} u(C_s), \qquad \text{s.t.} \tag{3.1.1}$$

$$A_{t+1} = R_{t+1} \left[A_t + w_t L_0 - C_t \right] \tag{3.1.2}$$

with interest factor $R_t \equiv 1+r_t$ and discount factor $\beta \equiv \frac{1}{1+\rho}$. The fact that the interest factor is also applied to the income and expenditure streams of the current period simply represents the assumed timing convention, i.e. those streams occur at the beginning of the period.[1] In recursive representation

$$U(A_t) = \max_{C_t}\ u(C_t) + \beta U(A_{t+1}), \qquad \text{s.t. (3.1.2).} \tag{3.1.3}$$

Define marginal life-time utility of wealth as $\lambda_t \equiv U'(A_t)$. Then the optimality condition is

$$C_t: \quad u'(C_t) = \beta R_{t+1} \lambda_{t+1}. \tag{3.1.4}$$

Differentiate (3.1.3) w.r.t. A_t to get the envelope condition:

$$A_t: \quad \lambda_t = \beta \lambda_{t+1} R_{t+1}. \tag{3.1.5}$$

Combine the optimality and envelope condition to get the typical Euler equation

$$u'(C_t) = \beta R_{t+1} u'(C_{t+1}) \qquad \Rightarrow \qquad C_{t+1} = \beta R_{t+1} C_t \tag{3.1.6}$$

for the special case of $u(\cdot) = \ln(\cdot)$. Recursive substitution of the Euler equation gives

$$C_s = \beta^{s-t} \prod_{u=t+1}^{s} R_u C_t. \tag{3.1.7}$$

[1] In contrast an 'end of period' timing convention would imply an intertemporal budget constraint of the following form $A_{t+1} = R_t A_t + w_t L_0 - C_t$. As long as the timing convention is consistently applied everywhere it does not make a difference for economic behavior.

We rewrite the budget constraint (3.1.2) by inserting recursively and use the no-Ponzi condition ($\lim_{T\to\infty} A_T \prod_{u=t+1}^{T}(R_u)^{-1} = 0$), i.e.

$$A_t = [C_t - w_t L_0] + \sum_{s=t+1}^{\infty} [C_s - w_s L_0] \prod_{u=t+1}^{s} (R_u)^{-1} \quad \text{or} \tag{3.1.8}$$

$$\mathcal{W}_t = A_t + H_t, \quad \text{where} \tag{3.1.9}$$

$$\mathcal{W}_t \equiv C_t + \sum_{s=t+1}^{\infty} C_s \prod_{u=t+1}^{s} (R_u)^{-1}, \quad H_t \equiv w_t L_0 + \sum_{s=t+1}^{\infty} w_s L_0 \prod_{u=t+1}^{s} (R_u)^{-1}.$$

Now insert the Euler equation (3.1.7) in the definition of total life-time wealth (3.1.9) to solve for the consumption function.

$$\begin{aligned}\mathcal{W}_t &= C_t + C_t \left(\sum_{s=t+1}^{\infty} \beta^{s-t} \prod_{u=t+1}^{s} R_u \prod_{u=t+1}^{s} (R_u)^{-1} \right) \\ &= C_t + C_t \left(\sum_{s=t+1}^{\infty} \beta^{s-t} \right) = C_t \left(\sum_{s=t}^{\infty} \beta^{s-t} \right) = C_t \cdot \frac{1}{1-\beta}.\end{aligned} \tag{3.1.10}$$

Hence, $\Omega \equiv 1/(1-\beta)$ is the inverse marginal propensity to consume which is constant for the special case of log-utility in the following consumption function

$$C_t = (1-\beta)\mathcal{W}_t = (1-\beta)(A_t + H_t) = \Omega^{-1}(A_t + H_t) \tag{3.1.11}$$

where H_t is forward looking and can be recursively written as

$$H_t = w_t L_0 + \frac{H_{t+1}}{R_{t+1}}. \tag{3.1.12}$$

3.1.3 Production

The production function is assumed to be homogeneous of degree one[2], e.g. of a simple Cobb-Douglas form

$$Y_t = f(K_t, L_t^D) = A^0 (K_t)^{\alpha} \left(L_t^D\right)^{1-\alpha}, \tag{3.1.13}$$

[2] A^0 denotes total factor productivity and is not to be confused with assets at time t A_t.

where L^D is labor demand which in equilibrium obviously has to equal supply L^S. Capital adjustment through investment is paid out of per-period earnings. Per-period profits (i.e. dividends) are therefore

$$\chi_t = Y_t - w_t L_t^D - I_t \tag{3.1.14}$$

The value of the firm is the discounted stream of per-period profits, i.e.

$$V_t = \chi_t + \frac{V_{t+1}}{R_{t+1}}, \tag{3.1.15}$$

and the law of motion for capital is

$$K_{t+1} = (1 - \delta^K)K_t + I_t. \tag{3.1.16}$$

The firm solves the following problem

$$V(K_t) = \max_{I_t, L_t^D} \chi_t + \frac{V(K_{t+1})}{R_{t+1}}. \tag{3.1.17}$$

Define the marginal benefit of an increase in capital $q_t \equiv V'(K_t)$. The optimality conditions are

$$I_t: \quad q_{t+1} = R_{t+1} \tag{3.1.18}$$

$$L_t^D: \quad Y_{L_t^D} = w_t. \tag{3.1.19}$$

Differentiate (3.1.17) w.r.t. K_t to get the envelope condition:

$$K_t: \quad q_t = Y_{K_t} + \frac{q_{t+1}}{R_{t+1}}(1 - \delta^K) \tag{3.1.20}$$

We briefly establish Hayashi (1982)'s theorem, which connects the firm value V and the capital stock K.

Theorem 3.1.1. *Hayashi's theorem. Firm value and capital stock fulfill the following relationship*

$$q_t K_t = V_t, \quad \forall t \tag{3.1.21}$$

Proof. Take the envelope condition for K_t (3.1.20) and multiply both sides by K_t and expand the right hand side by $\frac{q_{t+1}}{R_{t+1}} I_t$.

$$
\begin{aligned}
q_t K_t &= Y_{K_t} K_t + \frac{q_{t+1}}{R_{t+1}} \left[(1-\delta^K) K_t + I_t \right] - \frac{q_{t+1}}{R_{t+1}} I_t, \\
q_t K_t &= Y_{K_t} K_t + Y_{L_t^D} L_t^D - Y_{L_t^D} L_t^D - \frac{q_{t+1}}{R_{t+1}} I_t + \frac{q_{t+1}}{R_{t+1}} K_{t+1}, \\
q_t K_t &= Y_t - Y_{L_t^D} L_t^D - \frac{q_{t+1}}{R_{t+1}} I_t + \frac{q_{t+1}}{R_{t+1}} K_{t+1}, \\
q_t K_t &= \chi_t + \frac{q_{t+1}}{R_{t+1}} K_{t+1}.
\end{aligned}
\tag{3.1.22}
$$

From the first to the second line we used the law of motion (3.1.16) and expanded by the term $Y_{L_t^D} L_t^D$. From the second to the third line we used Euler's theorem and the linear homogeneity of the production function. From the third to the fourth line we inserted both optimality conditions and used the definition of per-period profits χ. Solving forward yields Hayashi (1982)'s result

$$
q_t K_t = \sum_{s=t}^{\infty} \chi_s \prod_{u=t+1}^{s} (R_u)^{-1} = V_t \tag{3.1.23}
$$

■

This shows that the shadow price q_t can be interpreted as Tobin's q, i.e. the ratio of firm value based on future earnings V and the capital replacement cost K. Using the law of motion for capital (3.1.16) and Hayashi's theorem evaluated at the optimality condition for investment, i.e. $q_{t+1} = R_{t+1}$ gives a simple relation for investment based on future profits:

$$
I_t = \frac{V_{t+1}}{R_{t+1}} - (1-\delta^K) K_t. \tag{3.1.24}
$$

3.1.4 Temporary Equilibrium

The task of finding a 'temporary' equilibrium is to find the market clearing prices for only one period (hence the expression 'temporary') for given values of forward looking variables H_{t+1} and V_{t+1}. In contrast the capital stock is known in t as it is predetermined by the law of motion through accumulation. First, knowing K_t we can calculate output as labor market clearing implies that $L_t^S = L_t^D \Rightarrow L_0 = L_t^D$

$$
Y_t = A^0 (K_t)^{\alpha} (L_0)^{1-\alpha}. \tag{3.1.25}
$$

Given output we can compute factor prices from the optimality and envelope conditions

$$R_t = \alpha\frac{Y_t}{K_t} + (1-\delta^K), \quad w_t = (1-\alpha)\frac{Y_t}{L_0}. \tag{3.1.26}$$

The last market that has to clear is the goods market

$$Y_t = C_t + I_t, \tag{3.1.27}$$

and we will use R_{t+1} as the clearing price.[3] As the household just invests in the firm this implies that all the financial wealth A coincides with V at the market clearing interest rate r. Consequently, the workers' consumption function is simply

$$C_t = (1-\beta)\mathcal{W}_t = \Omega^{-1}(V_t + H_t). \tag{3.1.28}$$

Insert the consumption (3.1.28) and the investment function (3.1.24) into (3.1.27).

$$Y_t = (1-\beta)\left[H_t + V_t\right] + \frac{V_{t+1}}{R_{t+1}} - (1-\delta^K)K_t \tag{3.1.29}$$

Now use (3.1.12), (3.1.15), (3.1.14) and (3.1.24) to get an implicit relationship of R_{t+1} as the only unknown

$$Y_t = (1-\beta)\left[\frac{H_{t+1}}{R_{t+1}} + Y_t + (1-\delta^K)K_t\right] + \frac{V_{t+1}}{R_{t+1}} - (1-\delta^K)K_t \tag{3.1.30}$$

In this simple case we can find an explicit solution for R_{t+1}

$$R_{t+1} = \frac{V_{t+1} + (1-\beta)\cdot H_{t+1}}{\beta\left[Y_t + (1-\delta^K)K_t\right]}. \tag{3.1.31}$$

Knowing R_{t+1} we can compute explicit solutions for all remaining unknowns: V_t, H_t, C_t, A_t, I_t and K_{t+1}.

[3] See section 3.1.5 on Walras' Law why we can do this.

3.1.5 Walras' Law

Define the following excess demands

$$\begin{aligned}
\text{assets}: \quad & \zeta_t^A = V_t - A_t && (3.1.32)\\
\text{labor}: \quad & \zeta_t^L = L_t^D - L_t^S && (3.1.33)\\
\text{goods}: \quad & \zeta_t^Y = C_t + I_t - Y_t && (3.1.34)
\end{aligned}$$

Rewrite (3.1.32) by inserting for V_t using (3.1.17) and eliminate χ_t by using (3.1.14) to get:

$$A_t = -\zeta_t^A + Y_t - w_t L_t^D - I_t + \frac{V_{t+1}}{R_{t+1}}. \tag{3.1.35}$$

Insert this expression in (3.1.2) to get

$$A_{t+1} - V_{t+1} = R_{t+1}\left[-\zeta_t^A + Y_t - I_t - C_t - w_t\left[L_t^D - L_t^S\right]\right]. \tag{3.1.36}$$

Use (3.1.32) at $t+1$, (3.1.33 and 3.1.34) to arrive at Walras' Law after some rearranging

$$\zeta_t^Y + w_t\zeta_t^L + \zeta_t^A - \frac{\zeta_{t+1}^A}{R_{t+1}} = 0. \tag{3.1.37}$$

This relationship has to hold even out of equilibrium and is very useful for bug-fixing the corresponding code.

3.1.6 Steady State

To have constant consumption given the Euler equation (3.1.6) it is clear that

$$\beta R = 1 \quad \Rightarrow \quad r = \rho. \tag{3.1.38}$$

Given exogenous labor supply and the marginal costs of capital we can simply compute capital usage according to the optimality condition for capital use (3.1.26)

$$K = L_0\left(\frac{\alpha A^0}{r + \delta^K}\right)^{\frac{1}{1-\alpha}}. \tag{3.1.39}$$

Given that we can calculate output and wages

$$Y = A^0 K^\alpha L_0^{1-\alpha} \quad \text{and} \quad w = (1-\alpha)\frac{Y}{L_0}. \tag{3.1.40}$$

Alternatively, output can be expressed directly in terms of L_0

$$Y = A^0 \left(\frac{\alpha A^0}{r + \delta^K} \right)^{\frac{\alpha}{1-\alpha}} L_0 \qquad \text{and} \qquad w = (1-\alpha) A^0 \left(\frac{\alpha A^0}{r + \delta^K} \right)^{\frac{\alpha}{1-\alpha}}, \tag{3.1.41}$$

which demonstrates that in this type of models long-run output is driven by labor supply, while long-run wages are independent of it. Steady state investment is where the capital stock in (3.1.16) stays constant

$$I = \delta^K K. \tag{3.1.42}$$

Dividends are given according to the definition (4.2.2)

$$\chi = Y - wL_0 - I. \tag{3.1.43}$$

Equilibrium values of human wealth and financial wealth are computed using (3.1.12) and (3.1.15)

$$H = \frac{1+r}{r} wL_0 \qquad \text{and} \qquad V = \frac{1+r}{r} \chi. \tag{3.1.44}$$

Finally, we can compute consumption

$$C = \Omega^{-1}(V + H). \tag{3.1.45}$$

Assets A have to equal V and can alternatively be computed from the steady state version of the intertemporal budget constraint (3.1.2), i.e.

$$A = \frac{1+r}{r} \left(C - wL_0 \right). \tag{3.1.46}$$

It is left as an exercise for the reader to confirm that $r = \rho$ is indeed the solution to the steady state version of equation (3.1.31).

3.1.7 Implementation

The codes implementing the model described in this section are called `Ramsey` and `Ramsey_simple`. The difference is that `Ramsey_simple` uses the explicit solution for the interest rate as given in (3.1.31). Using explicit solutions reduces the required computing time. In contrast `Ramsey` uses a numerical root finder to solve for the interest rate. This option is presented because with increasing complexity of a model using explicit solutions will be too cumbersome or simply impossible. Hence, the latter option is what we will have to use for all successively presented models. For the Ramsey model we focus on the description of the following two functions `TE()` and `path()`.

Code 2: Computing temporary equilibrium in a Ramsey model in `Ramsey`

```
% ----------------------
% TE.m
% ----------------------
% computes the temporary equilibrium for the global time 't'
% in:   interest rate
% out:  excess demand goods market
% ----------------------

function resid = TE(interest)

global t mpc
global Y Div w I V K H C edy r
global A_0 alpha L0 delta

r(t+1)  = interest;

% PRODUCTION AND INVESTMENT
Y(t)    = A_0*K(t)^alpha*L0^(1-alpha);
w(t)    = (1-alpha)*Y(t)/L0;
r(t)    = alpha*Y(t)/K(t)-delta;
I(t)    = V(t+1)/(1+r(t+1))-(1-delta)*K(t);
K(t+1)  = I(t)+(1-delta)*K(t);
Div(t)  = Y(t)-w(t)*L0-I(t);
V(t)    = Div(t)+V(t+1)/(1+r(t+1));

% CONSUMPTION AND SAVINGS
H(t)    = w(t)*L0+H(t+1)/(1+r(t+1));
C(t)    = mpc*(H(t)+V(t));

% EQUILIBRIUM
edy(t)  = I(t)+C(t)-Y(t);
resid   = edy(t);

end
```

The function `TE()` always computes temporary equilibrium and in general takes the following inputs and produces the following outputs. Depending on the model specification `TE()` finds the (temporary) equilibrium for n markets (or equilibrium conditions). In the case of our simply Ramsey model the clearing price for only one market has to be computed. By programming convention only the prices enter the function `TE()` (and also the function for computing a steady state `SS()`) as arguments. Other variables are passed on and edited globally.

```
ed = TE(prices)
```

`ed` .... vector of size n of excess demands
`prices`..... vector of size n of prices

Code 3: Computing a path of temporary equilibria in a Ramsey model

```
% ----------------------
% path.m
% ----------------------
% computes the path of temporary equilibrium until tend
% in:   guess, tstart (opt), tstop (opt)
% out:  actual
% ----------------------

function actual = path(guessin,tstart,tstop)

global t tend r
global V H

V       = guessin(:,1)';
H       = guessin(:,2)';

% COMPUTE ALL TEMPORARY EQUILIBRIA
if nargin < 3;
    tstop = tend;
end
if nargin > 1
    t = tstart;
else
    t = 1;
end
```

```

while t < tstop
    x0      = r(t); % inital guess
    [x1, retcode] = findroot(@TE,x0);
    if retcode~=0 error('Root finder did not converge!'); end
    t   = t+1;
end

actual = [V; H]';

end
```

`path()` simply computes a series of temporary equilibria and takes the following arguments

```
actual = path(guess,tstart,tstop)
```

`guess` ..... $T \times m$ matrix of the foresight variables
`tstart` .... an integer indicating the start of the path (default = 1)
`tstop` ..... an integer indicating the end of the path (default = T)

The argument `tstart` plays a role only if unanticipated reforms are to be simulated that are announced at a later point in time. See section 4.4.

3.1.8 Exercises

Exercises

Ex. 2 — Non-log-utility 1

Log-utility is a special case for which the marginal propensity to consume is constant. Work out the more general case where

$$u(C) = \frac{\sigma}{\sigma - 1}\left(C^{\frac{\sigma-1}{\sigma}} - 1\right), \quad \text{for } \sigma > 0,\ \sigma \neq 1.$$

Document and implement the necessary changes in the code `Ramsey`. Hint: Find a recursive expression for the marginal propensity to consume and treat it as any other foresight variable.

Ex. 3 — Non-log-utility 2

Simulate a shock to the capital stock where 20% is exogenously destroyed in period 1. Compare the recovery time for the cases $\sigma = 1$ versus $\sigma = 0.25$ and give a short interpretation.

3.2 Exogenous Growth and Detrending

In a realistic macroeconomic framework, especially one designed to do long run analysis, one has to incorporate growth components. We will do this in a very simple way by postulating that labor productivity grows at an exogenous rate g. Denote the level of technological progress as X. The current level is then simply

$$X_t = G^t X_0 \tag{3.2.1}$$

where G is the growth factor, i.e. $G = 1 + g$ and X_0 is the initial level of technological progress in $t = 0$. In a balanced growth setting all variables of the economy (output, consumption, investment, wages, capital stock, etc.)[4] grow at the exogenous rate g. For the analysis it is therefore convenient to detrend the economy by said factor G, i.e. everything is expressed in terms of labor efficiency units. Let us denote non-detrended variables with a tilde, e.g. $\tilde{K}$ in contrast to the detrended capital stock K, where latter is simply defined by $\tilde{K}_t = X_t K_t$, implicitly assuming $K_0 = \tilde{K}_0$. The typical law of motion for capital in non-detrended and detrended form is

$$\tilde{K}_{t+1} = (1 - \delta^K)\tilde{K}_t + \tilde{I}_t \quad \Leftrightarrow \quad GK_{t+1} = (1 - \delta^K)K_t + I_t. \tag{3.2.2}$$

The same procedure is applied to all the relevant difference equations, e.g. for assets A, firm value V, pension wealth P, etc. which are all linear. Expressing utility in terms of detrended variables is a little bit more sophisticated. Life time utility in period t is given as

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(\tilde{C}_t) = \sum_{s=t}^{\infty} \beta^{s-t} u(X_t C_t). \tag{3.2.3}$$

[4] A prerequisite for balanced growth is obviously a production function which is homogenous of degree one.

Assume $u(\cdot)$ of simple isoelastic form $u(XC) = (XC)^\rho$. Again $\rho = (\sigma - 1)/\sigma$. The problem is to get rid of X_t in the recursive formulation

$$U_t = u(X_t C_t) + \beta U_{t+1}, \tag{3.2.4}$$

and find a recursive expression that only depends on the growth factor G. We simply pull X_t out of the felicity functions, i.e. $u(X_t C_t) = (X_t)^\rho u(C_t)$ and $u(X_{t+1} C_{t+1}) = (X_t G)^\rho u(C_{t+1})$, etc. Define $V_t = U_t / X_t^\rho$, which for utility maximization is an equivalent formulation[5] compared to (3.2.4), to get

$$V_t = u(C_t) + \beta (G)^\rho V_{t+1}. \tag{3.2.5}$$

We finish this section by briefly looking at the solution to the intertemporal problem, i.e.

$$\begin{aligned} V_t = &\max_{C_t} \; u(C_t) + \beta (G)^\rho V_{t+1}, \quad \text{s.t.} \\ &GA_{t+1} = R_{t+1} \left[A_t + w_t L_t^S - C_t \right] \end{aligned} \tag{3.2.6}$$

The optimality and the envelope condition (again define the shadow price as $\lambda_t \equiv \partial V_t / \partial A_t$) are

$$C_t \;:\quad u'(C_t) = \beta (G)^{-\frac{1}{\sigma}} R_{t+1} \lambda_{t+1} \tag{3.2.7}$$

$$A_t \;:\quad \lambda_t = \beta (G)^{-\frac{1}{\sigma}} R_{t+1} \lambda_{t+1} \tag{3.2.8}$$

Hence, the Euler equation is

$$GC_{t+1} = [\beta R_{t+1}]^\sigma C_t, \tag{3.2.9}$$

which is consistent with the non-detrended version $\tilde{C}_{t+1} = [\beta R_{t+1}]^\sigma \tilde{C}_t$.

[5] Observe that the maximization of the often used recursive formulation $U_t = \left[\left(\tilde{C}_t \right)^\rho + \beta \left(U_{t+1} \right)^\rho \right]^{1/\rho}$ has the same solution as maximizing (3.2.4). Hence, detrending works analogously, i.e. $V_t = \left[(C_t)^\rho + \beta (G V_{t+1})^\rho \right]^{1/\rho}$.

Chapter 4

OLG - The Blanchard Model

4.1 Closed Economy

Summary

This section shows how to characterize the solution of a simple overlapping generations model in the Blanchard (1985)-spirit in a way that can be directly implemented and solved numerically on a computer. There is a mass of households which face an age-independent probability of death. It is shown that under certain assumptions the individual household decision rules can be **analytically aggregated** at the economy level, such that the computational costs of running simulations on a computer are virtually identical to the simply Ramsey model. The presented model is of the following very reduced form. Labor supply is fixed. The households' felicity functions are log. Households die at a constant rate. There is no population growth. Production is handled by a single representative firm that invests using retained earnings. The firm is owned by the households. There is no government. The corresponding code is `Blanchard`.

4.1.1 Description of the Economy

The economy exists of overlapping generations of households with identical members that supply labor (exogenously fixed at aggregate $L_t^S = L_0$) and own all production[1] facilities that produce a homogeneous good. There are

[1] Production works exactly as in the simple Ramsey model.

three markets: goods, labor and assets, with prices 1, w and r.

4.1.2 Households

Each period a fraction $1-\gamma$ of the households dies (irrespective of age). To keep track we denote variables with two subscripts: t is used as the time index and $v \leq t$ for the period of birth of the corresponding agent. This is of importance as asset level will differ for agents of different age, which implies that also consumption levels vary between those agents. Utility of being dead is normalized to 0.[2] Expected life-time utility can be recursively written as

$$U_{v,t} = u(C_{v,t}) + \beta \left[\gamma \cdot U_{v,t+1} + (1-\gamma) \cdot 0\right]. \tag{4.1.1}$$

where $\beta = \frac{1}{1+\rho}$ is the discount factor. There is an actuarially fair reverse life-insurance contract. The contract implies that if an individual dies, end of period assets (A^{end}) are ceased by the insurance company. If the individual survives she will get a premium (ϑA^{end}). Zero profit requires equating expected revenue and costs, i.e. $(1-\gamma)A^{end} = \gamma\vartheta A^{end}$, which implies that $\vartheta = \frac{1-\gamma}{\gamma}$ and that next period's assets of a surviving individual are: $A_{v,t+1} = (1+\vartheta)A^{end}_{v,t} = A^{end}_{v,t}/\gamma$ or more precisely

$$\gamma A_{v,t+1} = R_{t+1}\left[A_{v,t} + w_t \ell_0 - C_{v,t}\right]. \tag{4.1.2}$$

with interest factor $R_{t+1} \equiv 1 + r_{t+1}$. Every member in a household group maximizes life-time utility subject to this intertemporal budget constraint, i.e. in recursive representation

$$U(A_{v,t}) = \max_{C_{v,t}} \; u(C_{v,t}) + \beta\gamma U(A_{v,t+1}), \qquad \text{s.t. (4.1.2)}. \tag{4.1.3}$$

Define marginal life-time utility of wealth as $\lambda_{v,t} \equiv U'(A_{v,t})$. Then the optimality condition is

$$C_{v,t}: \quad u'(C_{v,t}) = \beta R_{t+1} \lambda_{v,t+1}. \tag{4.1.4}$$

[2] Normalizing utility of death to 0 in combination with log-utility can imply problems if consumption drops below 1 unit which makes individuals prefer death over life. As they have no active choice or influence on the probability of dying this assumption is harmless in our framework.

Differentiate (4.1.3) w.r.t. $A_{v,t}$ to get the envelope condition:

$$A_{v,t}: \quad \lambda_{v,t} = \beta \lambda_{v,t+1} R_{t+1}. \tag{4.1.5}$$

Combine the optimality and envelope condition to get the typical Euler equation

$$u'(C_{v,t}) = \beta R_{t+1} u'(C_{v,t+1}) \quad \Rightarrow \quad C_{v,t+1} = \beta R_{t+1} C_{v,t} \tag{4.1.6}$$

for the special case of $u(\cdot) = \ln(\cdot)$. Recursive substitution of the Euler equation gives

$$C_{v,s} = \beta^{s-t} \prod_{u=t+1}^{s} R_u C_{v,t}. \tag{4.1.7}$$

We rewrite the budget constraint (4.1.2) by inserting recursively and using the no-Ponzi condition ($\lim_{T\to\infty} A_{v,T} \prod_{u=t+1}^{T} (R_u)^{-1} = 0$), i.e.

$$A_{v,t} = [C_{v,t} - w_{v,t}\ell_0] + \sum_{s=t+1}^{\infty} [C_{v,s} - w_s \ell_0] \prod_{u=t+1}^{s} \frac{\gamma}{R_u} \quad \text{or} \tag{4.1.8}$$

$$\mathcal{W}_{v,t} = A_{v,t} + H_{v,t}, \quad \text{where} \tag{4.1.9}$$

$$\mathcal{W}_{v,t} \equiv C_{v,t} + \sum_{s=t+1}^{\infty} C_{v,s} \prod_{u=t+1}^{s} \frac{\gamma}{R_u}, \quad H_{v,t} \equiv w_t \ell_0 + \sum_{s=t+1}^{\infty} w_s \ell_0 \prod_{u=t+1}^{s} \frac{\gamma}{R_u}.$$

Now insert the Euler equation (4.1.7) in the definition of total life-time wealth (4.1.9) to solve for the consumption function.

$$\begin{aligned} \mathcal{W}_{v,t} &= C_{v,t} + C_{v,t} \left(\sum_{s=t+1}^{\infty} \beta^{s-t} \prod_{u=t+1}^{s} R_u \prod_{u=t+1}^{s} \frac{\gamma}{R_u} \right) \\ &= C_{v,t} + C_{v,t} \left(\sum_{s=t+1}^{\infty} \beta^{s-t} \right) = C_{v,t} \left(\sum_{s=t}^{\infty} \beta^{s-t} \right) = C_{v,t} \cdot \frac{1}{1-\beta\gamma}. \end{aligned} \tag{4.1.10}$$

Hence, the consumption function is where $\Omega \equiv 1 - \beta\gamma$ is the inverse marginal propensity to consume which is constant for the special case of log-utility,

$$C_{v,t} = (1-\beta\gamma)\mathcal{W}_{v,t} = \Omega^{-1}(A_{v,t} + H_{v,t}) \tag{4.1.11}$$

where $A_{v,t}$ and $H_{v,t}$ are forward looking and can be recursively written as

$$A_{v,t} = C_{v,t} - w_t \ell_0 + \gamma \frac{A_{v,t+1}}{R_{t+1}}, \tag{4.1.12}$$

$$H_{v,t} = w_t \ell_0 + \gamma \frac{H_{v,t+1}}{R_{t+1}}. \tag{4.1.13}$$

4.1.3 Aggregation

The size of a population group born at v at time t is denoted $N_{v,t}$. As a constant fraction of every group dies every period we have to following law of motion for the size of group v

$$N_{v,t+1} = \gamma N_{v,t}. \tag{4.1.14}$$

At the same time a new cohort is born when moving from t to $t+1$, namely $N_{t+1,t+1}$. A simple accounting identity is[3]

$$N_t \equiv \sum_{v=-\infty}^{t} N_{v,t}. \tag{4.1.15}$$

We abstract from aggregate population growth[4] which means that the total number of deaths $\sum_{v=-\infty}^{t}(1-\gamma)N_{v,t} = (1-\gamma)N_t$ has to equal the total number of births $N_{t+1,t+1}$. Summing the group specific laws of motion (4.1.14) for v between $-\infty$ and t and adding $N_{t+1,t+1}$ to both sides gives the aggregate law of motion

$$N_{t+1} = N_{t+1,t+1} + \gamma N_t. \tag{4.1.16}$$

Inserting the equality of number of deaths and births $(1-\gamma)N_t = N_{t+1,t+1}$ reveals that indeed we have $N_{t+1} = N_t \equiv N$. All other per capita variables, call them $X_{v,t}$, are aggregated similarly using the population shares as weights, e.g.

$$X_t = \sum_{v=-\infty}^{t} X_{v,t} N_{v,t}. \tag{4.1.17}$$

If per capita variables do not differ by age group v the aggregation is even simpler: $X_{v,t} = x_t \Rightarrow X_t = x_t N$. We assumed that exogenous labor supply is

[3] $\sum_{v=-\infty}^{t}$ is a notational abbreviation for $\lim_{k\to-\infty}\sum_{v=k}^{t}$.

[4] See section 4.3 for details on demographic change in a Blanchard model.

equally distributed among the population[5], i.e. ℓ_0. Given the 'no population growth' assumption, aggregate labor supply $L_0 = \ell_0 N$ is also exogenous as claimed before. Consequently, also human wealth does not differ between members of different age groups and we have $H_{v,t} = h_t$ and therefore $H_t = h_t N$. Aggregating human wealth simply works by multiplying with N

$$H_t = w_t L_0 + \gamma \frac{H_{t+1}}{R_{t+1}}. \tag{4.1.18}$$

However, note that (4.1.18) is only true in case of constant population N. See section 4.3 for the case of demographic change. To get the aggregate consumption function we multiply by $N_{v,t}$ and sum over all individual consumption functions (4.1.11)

$$C_t = \Omega^{-1} \left[A_t + H_t \right]. \tag{4.1.19}$$

We now aggregate all resource constraints (4.1.2) to get

$$\gamma \sum_{v=-\infty}^{t} A_{v,t+1} N_{v,t} = R_{t+1} \left[A_t + w_t L_0 - C_t \right]. \tag{4.1.20}$$

Note that the left hand side can be rewritten using (4.1.14) and the by adding $A_{t+1,t+1} N_{t+1,t+1} = 0$, which is zero because newborns do not possess any assets, we have

$$\gamma \sum_{v=-\infty}^{t} A_{v,t+1} N_{v,t} = \sum_{v=-\infty}^{t} A_{v,t+1} N_{v,t+1} = \sum_{v=-\infty}^{t+1} A_{v,t+1} N_{v,t+1} - A_{t+1,t+1} N_{t+1,t+1}$$
$$= \sum_{v=-\infty}^{t+1} A_{v,t+1} N_{v,t+1} = A_{t+1}. \tag{4.1.21}$$

Hence the aggregate asset equation does not contain γ and reads as follows

$$A_{t+1} = R_{t+1} \left[A_t + w_t L_0 - C_t \right]. \tag{4.1.22}$$

[5]... as are individual skills which justifies a common wage w_t.

We now try to aggregate the Euler equation (3.1.6). It is not necessary or particularly helpful for the following calculations but gives some additional insights. First multiply (3.1.6) by $N_{v,t}$ and then sum from $v = t$ to $-\infty$.

$$\sum_{v=-\infty}^{t} C_{v,t+1} N_{v,t} = \beta R_{t+1} \sum_{v=-\infty}^{t} C_{v,t} N_{v,t}. \tag{4.1.23}$$

The right hand sight gives $\beta R_{t+1} C_t$ by the aggregation definition. Insert (4.1.14) on the left hand side and multiply by γ to get

$$\sum_{v=-\infty}^{t} C_{v,t+1} N_{v,t+1} = \gamma \beta R_{t+1} C_t. \tag{4.1.24}$$

Now expand by $C_{t+1,t+1} N_{t+1,t+1} - C_{t+1,t+1} N_{t+1,t+1}$ and note that $N_{t+1,t+1} = (1-\gamma)N$. Rearrange to get

$$C_{t+1} = \gamma \beta R_{t+1} C_t + N(1-\gamma) C_{t+1,t+1}. \tag{4.1.25}$$

This is a nice expression for interpretation. First, note that because the consumption of the newborns is bigger than zero, i.e. $C_{t+1,t+1} > 0$, we must have $\gamma \beta R_{t+1} < 1$ for stable aggregate consumption. Interestingly this does not put a restriction on whether $\rho > r$, $\rho = r$ or $\rho < r$. Second, if we let $\gamma \to 1$, i.e. no one dies and no one is born, we arrive at the typical Ramsey-kind Euler equation

$$C_{t+1} = \beta R_{t+1} C_t. \tag{4.1.26}$$

4.1.4 Production

Production is identical to the closed economy case in the Ramsey model, see section 3.1.3.

4.1.5 Temporary Equilibrium

Only some of the defining temporary equilibrium conditions change in comparison to the closed economy case in the Ramsey model, see section 3.1.4. We proceed in the same order. Output Y_t and factor prices R_t and w_t are computed as before. The clearing condition for the goods market is again

$$Y_t = C_t + I_t. \tag{4.1.27}$$

Insert the consumption (4.1.19) and the investment function (3.1.24) into (4.1.27).

$$Y_t = (1 - \beta\gamma)\left[H_t + V_t\right] + \frac{V_{t+1}}{R_{t+1}} - (1 - \delta^K)K_t \tag{4.1.28}$$

Now use (4.1.18), (3.1.15), (3.1.14) and (3.1.24) to get an implicit relationship of R_{t+1} as the only unknown

$$Y_t = (1 - \beta\gamma)\left[\gamma\frac{H_{t+1}}{R_{t+1}} + Y_t + (1 - \delta^K)K_t\right] + \frac{V_{t+1}}{R_{t+1}} - (1 - \delta^K)K_t \tag{4.1.29}$$

In this simple case we can find an explicit solution for R_{t+1}

$$R_{t+1} = \frac{V_{t+1} + (1 - \beta\gamma) \cdot \gamma H_{t+1}}{\beta\gamma\left[Y_t + (1 - \delta^K)K_t\right]}. \tag{4.1.30}$$

Knowing R_{t+1} we can compute explicit solutions for all remaining unknowns: V_t, H_t, C_t, A_t, I_t and K_{t+1}.

4.1.6 Walras' Law

The derivation of Walras' Law is identical to the closed economy case in the Ramsey model, see section 3.1.5.

4.1.7 Steady State

The calculation of the steady state follows a very similar logic as for the closed economy case in the Ramsey model, see (3.1.6), with the following exceptions. Before looking at the steady state of the Blanchard model in more detail note the following important remark about leaving the realm of working with a single representative household. In the steady state we require *aggregate* variables to be constant. Individual consumption and asset paths might be non-constant even in steady state. For example a monotonically increasing ($r > \rho$) or decreasing ($r < \rho$) individual consumption profile over the life-cycle is possible now as life-cycles end at some point. Hence, we cannot conjecture from the individual Euler equation (4.1.6) that $r = \rho$ has to be true to have constant aggregate consumption. Instead we have to look at the aggregate Euler equation (4.1.25). Equilibrium interest rate r cannot be pinned down by the preference parameter ρ. Instead r is determined as the solution to the steady state equation of (4.1.30).[6]

$$R = \frac{V + (1-\beta\gamma)\cdot\gamma H}{\beta\gamma\left[Y + (1-\delta^K)K\right]}. \tag{4.1.31}$$

Further, the two steady state equations for aggregate human wealth H and aggregate consumption C differ from the Ramsey model

$$C = \Omega^{-1}\left[V + H\right], \qquad \text{with} \;\; \Omega = 1/(1-\beta\gamma) \tag{4.1.32}$$

$$H = \frac{1+r}{r + (1-\gamma)} w L_0. \tag{4.1.33}$$

Inserting accordingly into (4.1.31) gives a single equation depending on known parameters and one unknown r which can be solved for numerically.

4.2 Small Open Economy

Summary

This section extends the previously presented Blanchard model to a **small open economy** setting, all other assumptions unchanged. Hence, the interest rate is taken as given and asset markets clear through adjustments of foreign assets. In order to replicated realistic changes in the capital stock adjustment costs of quadratic form are introduced. The presented model is of the following very reduced form. Labor supply is fixed. The households' felicity functions are log. Households die at a constant rate. There is no population growth. Production is handled by a single representative firm that invests using retained earnings. Capital adjustment is subject to costs. The firm is owned by the households. There is no government. The corresponding code is `Blanchard_open`.

[6] In the Ramsey model both ways of determining r are equivalent.

4.2.1 Description of the Economy

The economy exists of overlapping generations of households with identical members that supply labor (exogenously fixed at aggregate $L_t^S = L_0$) and own all production facilities that produce a homogeneous good at price 1. The economy is assumed to be small and open, i.e. a world interest rate of r is taken as given. This implies that excess assets are simply invested abroad as foreign asset demand is inelastic. Consequently, access demand for goods is satisfied by importing the homogeneous good. By assumption capital accumulation is subject to capital-adjustment costs. Without adjustment costs the representative firm would react to every shock by adjusting the capital stock optimally in only a single period as the rental price for capital r is constant.

4.2.2 Households

The description of households and their decisions is identical to the closed economy case, see section 4.1.2.

4.2.3 Aggregation

The description of aggregation is identical to the closed economy case, see section 4.1.3.

4.2.4 Production

Production works almost identical to the models described above, see section 3.1.3. Again, we only highlight the necessary changes starting from the closed economy case. The important difference is in the assumption of capital adjustment costs J_t which are linear homogeneous in the inputs I_t and K_t. Specifically we assume the following functional form

$$J(I_t, K_t) = \frac{1}{2}\psi K_t \left(\frac{I_t}{K_t} - \delta^K\right)^2, \tag{4.2.1}$$

which is convexly increasing in I_t and is normalized to fulfill $J = 0$ in steady state. Dividends are revenue minus labor costs, investment and installation costs

$$\chi_t = Y_t - w_t L_0 - I_t - J_t. \tag{4.2.2}$$

As before the firm solves the following problem

$$V(K_t) = \max_{I_t, L_t^D} \ \chi_t + \frac{V(K_{t+1})}{R_{t+1}}. \tag{4.2.3}$$

Define the marginal benefit of an increase in capital $q_t \equiv V'(K_t)$. The optimality conditions are

$$I_t: \quad q_{t+1} = R_{t+1}\left(1 + J_{I_t}\right) \tag{4.2.4}$$

$$L_t^D: \quad Y_{L_t^D} = w_t. \tag{4.2.5}$$

Differentiate (4.2.3) w.r.t. K_t to get the envelope condition:

$$K_t: \quad q_t = Y_{K_t} - J_{K_t} + \frac{q_{t+1}}{R_{t+1}}(1 - \delta^K) \tag{4.2.6}$$

We briefly establish Hayashi (1982)'s theorem 3.1.1, which connects the firm value V and the capital stock K is also true in the case of linear homogeneous capital adjustment costs.

Proof. Take the envelope condition for K_t (4.2.6) and multiply both sides by K_t and expand the right hand side by $\frac{q_{t+1}}{R_{t+1}} I_t$.

$$\begin{aligned}
q_t K_t &= Y_{K_t} K_t - J_{K_t} K_t + \frac{q_{t+1}}{R_{t+1}}\left[(1 - \delta^K)K_t + I_t\right] - \frac{q_{t+1}}{R_{t+1}} I_t, \\
q_t K_t &= Y_{K_t} K_t + Y_{L_t^D} L_t^D - Y_{L_t^D} L_t^D - J_{K_t} K_t - \frac{q_{t+1}}{R_{t+1}} I_t + \frac{q_{t+1}}{R_{t+1}} K_{t+1}, \\
q_t K_t &= Y_t - Y_{L_t^D} L_t^D - J_{K_t} K_t - J_{I_t} I_t + J_{I_t} I_t - \frac{q_{t+1}}{R_{t+1}} I_t + \frac{q_{t+1}}{R_{t+1}} K_{t+1}, \\
q_t K_t &= Y_t - w_t L_t^D - J_t - I_t + \frac{q_{t+1}}{R_{t+1}} K_{t+1}, \\
q_t K_t &= \chi_t + \frac{q_{t+1}}{R_{t+1}} K_{t+1}.
\end{aligned} \tag{4.2.7}$$

From the first to the second line we used the law of motion (3.1.16) and expanded by the term $Y_{L_t^D} L_t^D$. From the second to the third line we used Euler's theorem and the linear homogeneity of the production function. We further expand by the term $J_{I_t} I_t$. From the third to the fourth line we inserted both optimality conditions and used Euler's theorem and the linear homogeneity of the capital adjustment cost function. From the fourth to the fifth line we used the definition of per-period profits χ. Solving forward yields Hayashi (1982)'s result

$$q_t K_t = \sum_{s=t}^{\infty} \chi_s \prod_{u=t+1}^{s} (R_u)^{-1} = V_t \tag{4.2.8}$$

■

Using the law of motion for capital (3.1.16) and Hayashi's theorem evaluated at the optimality condition for investment, i.e. $q_{t+1} = R_{t+1}(1 + J_{I_t})$ gives an implicit relation for investment:

$$I_t = \frac{V_{t+1}}{R_{t+1}(1 + J_{I_t})} - (1 - \delta^K)K_t. \tag{4.2.9}$$

Using our assumed functional form for J_t we can solve explicitly for I_t as the solution to the following quadratic equation:

$$\begin{aligned} &\left[\frac{\psi}{K_t}\right] \cdot I_t^2 + \left[(1 - \delta^K \psi) + (1 - \delta^K)\psi\right] \cdot I_t \\ &\quad + \left[(1 - \delta^K)(1 - \delta^K \psi)K_t - \frac{V_{t+1}}{R_{t+1}}\right] = 0. \end{aligned} \tag{4.2.10}$$

4.2.5 Foreign Assets and the Trade Balance

As the economy is small and open the interest rate r_t is given. Excess supply of assets is simply absorbed by the net foreign asset position D_t^F. At any time the portfolio identity has to hold

$$A_t = D_t^F + V_t, \tag{4.2.11}$$

i.e. households can invest their assets into foreign D_t^F or domestic assets V_t. Non-arbitrage implies that they are indifferent between both assets. The size of V_t is therefore simply determined by asset demand for the given interest rate r_t. Foreign assets D^F then just absorb the excess supply from households. The law of motion for foreign assets D^F can therefore be written as

$$D_{t+1}^F = R_{t+1}\left[D_t^F + TB_t\right], \quad \text{where} \quad TB_t \equiv Y_t - C_t - I_t - J_t, \tag{4.2.12}$$

where TB is the trade balance.

Proof. Use this expression in the aggregate asset equation (4.1.22) and use (3.1.15) to substitute for V_t and (4.2.2) to eliminate χ_t:

$$\frac{A_{t+1}}{R_{t+1}} = D_t^F + Y_t - C_t - I_t - J_t + \frac{V_{t+1}}{R_{t+1}}. \tag{4.2.13}$$

Use (4.2.11) at $t+1$ again to arrive at equation (4.2.12). ∎

4.2.6 Temporary Equilibrium

Characterizing and computing the temporary equilibrium works very similar to the closed economy case in the Blanchard model, see section 4.1.5. We just highlight the differences. In contrast to before the interest rate r_t is known as it is exogenous. The same is true for the current capital stock K_t and aggregate foreign assets D_t^F as they are predetermined. Output Y_t and wage w_t are computed as before. Given our guess for V_{t+1} we get investment by using (4.2.9) and solving the quadratic equation

$$I_t = \frac{V_{t+1}}{R_{t+1}(1 + J_{I_t})} - (1 - \delta^K)K_t \Rightarrow I_t. \tag{4.2.14}$$

Knowing K_t and I_t one can compute the corresponding adjustment costs J_t (4.2.1) and next periods capital stock K_{t+1} (3.1.16). Next, one can calculate dividends χ_t (4.2.2), aggregate human wealth H_t (4.1.18) and firm value V_t (3.1.15) and consequently consumption C_t by inserting the portfolio identity (4.2.11) into the aggregate consumption function (4.1.19)

$$C_t = \Omega^{-1}\left[D_t^F + V_t + H_t\right]. \tag{4.2.15}$$

Next we can infer trade balance and next period's foreign assets using (4.2.12).

4.2.7 Walras' Law

Define the following excess demands

$$\text{assets}: \quad \zeta_t^A = V_t + D_t^F - A_t \tag{4.2.16}$$

$$\text{labor}: \quad \zeta_t^L = L_t^D - L_t^S \tag{4.2.17}$$

$$\text{goods}: \quad \zeta_t^Y = C_t + I_t + J_t + TB_t - Y_t \tag{4.2.18}$$

Rewrite (4.2.16) by inserting for V_t using (3.1.15) and eliminate χ_t by using (4.2.2) to get:

$$A_t = -\zeta_t^A + D_t^F + Y_t - w_t L_t^D - I_t - J_t + \frac{V_{t+1}}{R_{t+1}}. \tag{4.2.19}$$

Insert this expression in the aggregate intertemporal budget constraint (4.1.22) to get

$$\frac{A_{t+1} - V_{t+1}}{R_{t+1}} = -\zeta_t^A + D_t^F + Y_t - I_t - J_t - C_t - w_t \left[L_t^D - L_t^S\right]. \tag{4.2.20}$$

Use (4.2.16) at $t+1$, (4.2.17 and 4.2.18) to arrive

$$\frac{D_{t+1}^F - \zeta_{t+1}^A}{R_{t+1}} = -\zeta_t^A + D_t^F + TB_t - \zeta_t^Y - w_t \zeta_t^L. \tag{4.2.21}$$

Insert the law of motion for foreign assets (4.2.12) to arrive at Walras' Law

$$\zeta_t^Y + w_t \zeta_t^L + \zeta_t^A - \frac{\zeta_{t+1}^A}{R_{t+1}} = 0. \tag{4.2.22}$$

4.2.8 Steady State

Again, computing the steady state works analogously to the sections (3.1.6) and (4.1.7) before. Interest rate r is fixed exogenously. Following section (3.1.6) we can compute K, Y, w and I. Evaluating capital adjustment costs at $I = \delta^K K$ reveals that they are zero in steady state, i.e. $J = 0$. Steady state dividends are given as

$$\chi = Y - wL_0 - I - J. \tag{4.2.23}$$

Now we can compute H and V following section (4.1.7). This leaves the following two relationships which have to be determined simultaneously by finding C and D^F

$$C = (1-\beta\gamma)\left[V + D^F + H\right], \tag{4.2.24}$$

$$D^F = \frac{1+r}{r}\left[C + I + J - Y\right]. \tag{4.2.25}$$

Once C and D^F are known one can compute aggregate assets A either using the portfolio identity $A = V + D^F$ or the aggregate intertemporal budget constraint

$$A = \frac{1+r}{r}\left(C - wL_0\right). \tag{4.2.26}$$

4.3 Demographic Change

4.3.1 Time varying demographic parameters

In this section we look at demographic transitions. We therefore have to relax two assumptions we made in the sections before:

- Survival rates are time dependent, i.e. the survival rate at end of period t is γ_{t+1}. Hence, we assume an exogenous sequence of γ_t that can differ for t but satisfy $\lim_{t\to\infty}\gamma_t = \gamma$.
- The number of newborns is time dependent and not necessarily connected to the number of deaths, i.e. the number of newborns at the beginning of period t is given as NB_t. We assume an exogenous sequence of NB_t that satisfies $\lim_{t\to\infty} NB_t = NB$

The limiting conditions imply that a stationary demographic distribution exists. The assumptions also imply that population size N_t can vary during transition but will eventually converge to some number N. Hence, the individual law of motion for population size per age group v is

$$N_{v,t+1} = \gamma_{t+1} N_{v,t}. \tag{4.3.1}$$

For the aggregate population it is

$$N_{t+1} = \gamma_{t+1} N_t + NB_{t+1}. \tag{4.3.2}$$

Note that with varying population also aggregate labor supply is time dependent, i.e. $L_t^S = \ell_0 N_t$. The household problem is written as

$$U(A_{v,t}) = \max_{C_{v,t}} \; u(C_{v,t}) + \beta\gamma_{t+1} U(A_{v,t+1}), \qquad \text{s.t. } . \tag{4.3.3}$$

$$\gamma_{t+1} A_{v,t+1} = R_{t+1} \left[A_{v,t} + w_t \ell_0 - C_{v,t} \right]. \tag{4.3.4}$$

Assume $u(x) = \frac{\sigma}{\sigma-1}\left(x^{\frac{\sigma-1}{\sigma}} - 1\right)$ such that $u'(x) = x^{-\frac{1}{\sigma}}$. Following the step of section 4.1.2 we can write the Euler equation as

$$C_{v,t+1} = \left(\beta R_{t+1}\right)^{\sigma} C_{v,t}. \tag{4.3.5}$$

Further, the budget constraint (4.3.4) and total wealth can be written as

$$A_{v,t} = \left[C_{v,t} - w_{v,t}\ell_0\right] + \sum_{s=t+1}^{\infty} \left[C_{v,s} - w_s \ell_0\right] \prod_{u=t+1}^{s} \frac{\gamma_u}{R_u} \quad \text{or} \tag{4.3.6}$$

$$\mathcal{W}_{v,t} = A_{v,t} + H_{v,t}, \quad \text{where} \tag{4.3.7}$$

$$\mathcal{W}_{v,t} \equiv C_{v,t} + \sum_{s=t+1}^{\infty} C_{v,s} \prod_{u=t+1}^{s} \frac{\gamma_u}{R_u}, \quad H_{v,t} \equiv w_t \ell_0 + \sum_{s=t+1}^{\infty} w_s \ell_0 \prod_{u=t+1}^{s} \frac{\gamma_u}{R_u}.$$

Insert the Euler equation consecutively to solve for the consumption function.

$$\begin{aligned} \mathcal{W}_{v,t} &= C_{v,t} + C_{v,t}\left(\sum_{s=t+1}^{\infty} \beta^{(s-t)\sigma} \prod_{u=t+1}^{s} (R_u)^{\sigma} \prod_{u=t+1}^{s} \frac{\gamma_u}{R_u} \right) \\ &= C_{v,t} + C_{v,t}\left(\sum_{s=t+1}^{\infty} \beta^{(s-t)\sigma} \prod_{u=t+1}^{s} (R_u)^{\sigma-1} \gamma_u \right) \qquad (4.3.8) \\ &= C_{v,t}\Omega_t, \qquad \text{where} \qquad (4.3.9) \end{aligned}$$

$$\Omega_t = 1 + \left(\sum_{s=t+1}^{\infty} \beta^{(s-t)\sigma} \prod_{u=t+1}^{s} (R_u)^{\sigma-1} \gamma_u \right) \tag{4.3.10}$$

Hence, the consumption function is

$$C_{v,t} = (\Omega_t)^{-1} \mathcal{W}_{v,t} = (\Omega_t)^{-1} \left(A_{v,t} + H_{v,t}\right) \tag{4.3.11}$$

where $A_{v,t}$, $H_{v,t}$ and the inverse marginal propensity Ω_t to consume are forward looking and can be recursively written as

$$\Omega_t = 1 + \beta^\sigma \left(R_{t+1}\right)^{\sigma-1} \gamma_{t+1}\Omega_{t+1}, \tag{4.3.12}$$

$$A_{v,t} = C_{v,t} - w_t\ell_0 + \gamma_{t+1}\frac{A_{v,t+1}}{R_{t+1}}, \tag{4.3.13}$$

$$H_{v,t} = w_t\ell_0 + \gamma_{t+1}\frac{H_{v,t+1}}{R_{t+1}}. \tag{4.3.14}$$

4.3.2 Aggregation

While aggregation of all static relationships and also assets works like before in section (4.1.3) we have to be cautious for other difference equations like (4.3.14). Solving forward reveals that $H_{v,t}$, the human wealth for a household of age v at time t, is independent of v. This implies that aggregate human wealth is $H_t = \sum_{v=-\infty}^{t} H_{v,t}N_{v,t} = H_{v,t}N_t$. Multiplying (4.3.14) by N_t and using $L_t^S = \ell_0 N_t$ gives

$$H_t = w_t L_t^S + \gamma_{t+1}\frac{H_{t+1}(N_t/N_{t+1})}{R_{t+1}} \quad \text{or} \tag{4.3.15}$$

This coincides with (4.1.18) only if $N = N_t = N_{t+1}$. This shows that this type of aggregation problems always have to be tackled using an per-capita interpretation. The Blanchard model with time-varying demographic parameters is implemented in `Blanchard_demo`.

4.3.3 Exercises

Exercises

For the following exercises simulate and *explain* the reaction (medium and long-run) of various macro variables: population size, labor supply, wage rate, capital stock, output, consumption, and foreign assets. Differentiate between absolute and per capita effects. Focus in the interpretation especially an the qualitative differences between both exercises. Use the code `Blanchard_demo`.

Ex. 4 — Higher fertility
Linearly increase the number of newborns by 10% over the first 30 years and keep it at that level afterward.

Ex. 5 — Longevity
Linearly increase the survival rate over the first 30 years (and keep it constant afterward) such that life expectancy for newborns rises by 5 years.

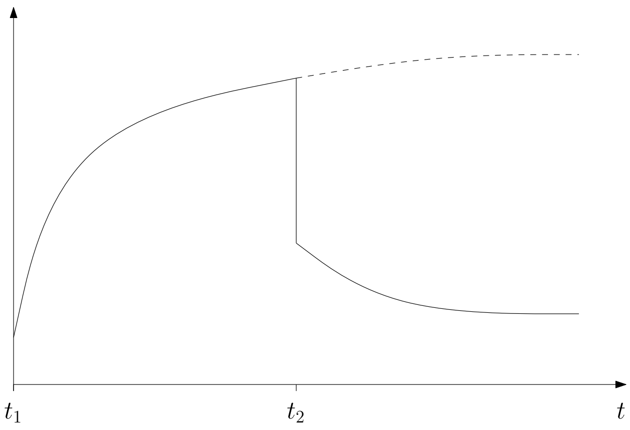
4.4 Timing of Unanticipated Shocks

Summary

This short section briefly explains how several unanticipated shocks with different timing can be simulated. The corresponding code is `Blanchard_unanticipated`.

Previously we have dealt with unanticipated shocks which occur when the economy is in a steady state. The time of the announcement was set to $t = 1$, i.e. households and firms with perfect foresight started to react to a policy change in $t = 1$ (even if the reform is only introduced in $t > 1$). However, how do we simulate shocks that are announced while in transition because of a previous shock? For example demographic change is an ongoing process. Simulating a pension reform to address the effects of aging starting from a steady state entails the problem that at the announcement of the reform the demography was not in equilibrium. Conceptually, this implies the same problems as simulating two different tax reforms, one announced in $t = 1$ and one in $t = 5$. It is important to emphasize the difference to a situation where two tax reforms, one starting in $t = 1$ and the second being effective only after $t = 5$, are introduced but both are announced at $t = 1$. We deal with this in the following way. Assume there are M reforms with M different announcement dates t_m. We sort them in ascending order in time and normalize the earliest announcement date to $t_1 = 1 < t_2 < ... < t_M$. We now have to carry out M full simulation iterations. We first start with reform 1 and solve for the time-consistent path from t_1 to the last period T. In the second simulation we introduce reform 2 but we take all values from the first simulation from t_1 to t_2 and compute the time-consistent path from t_2 to T, and so on. Figure 4.4 illustrates this technique for two reforms. The implementation in the code works as shown in the code extract 4 of the file `runme.m`.

Figure 4.4.1: The evolution of some variable X for computing two differently timed shocks in a Blanchard model



Code 4: Computing two differently timed shocks in a Blanchard model - Snippet from `runme.m`

```
% COMPUTE INITIAL STEADY STATE
ISS;

% LOAD FIRST REFORM
reform1;

% COMPUTE FINAL STEADY STATE
FSS;

% SOLVE FOR TRANSITIONS STARTING AT t=1
[actual, retcode] = gft(guess,jacob);

% LOAD SECOND REFORM
tstart          = 40;
reform2;

% COMPUTE FINAL STEADY STATE
FSS;

% SOLVE FOR TRANSITIONS STARTING AT tstart
[actual, retcode] = gft(guess,jacob,1,1,tstart);
```

4.5 Endogenous Labor Supply

This section briefly sketches the necessary changes involved for endogenizing labor supply first along the intensive margin (i.e. hours) and then along the extensive margin (i.e. participation). We only highlight the adaption of the primitives of the model and the optimality conditions. A full derivation is provided in section 4.6 where in addition taxes are introduced.

4.5.1 Intensive Margin

We assume that a household's consumption bundle $Q_{v,t}$ consists of goods consumption and leisure captured by disutility of labor in an additively separable way, i.e. neglecting income effects: $Q_{v,t} = C_{v,t} - \varphi(\ell_{v,t})$. Instantaneous utility is simply $u(Q)$. Hence, the problem of a household born at v is

$$U(A_{v,t}) = \max_{C_{v,t},\ell_{v,t}} u(Q_{v,t}) + \beta\gamma U(A_{v,t+1}), \qquad \text{s.t.} \tag{4.5.1}$$

$$\gamma_{t+1}A_{v,t+1} = R_{t+1}\left[A_{v,t} + w_t\ell_{v,t}\theta_t - C_{v,t}\right] \qquad \text{and} \tag{4.5.2}$$

$$Q_{v,t} = C_{v,t} - \varphi(\ell_{v,t}). \tag{4.5.3}$$

θ_t is assumed to be an exogenous parameter for labor productivity which implies that w_t is interpreted as wage rate per efficiency unit. The two optimality and the envelope conditions using the usual definition of $\lambda_{v,t}$ are

$$C_{v,t}: \quad u'(Q_{v,t}) = \beta R_{t+1}\lambda_{v,t+1}, \tag{4.5.4}$$

$$\ell_{v,t}: \quad u'(Q_{v,t})\varphi'(\ell_{v,t}) = \beta R_{t+1}\lambda_{v,t+1}w_t\theta_t \tag{4.5.5}$$

$$A_{v,t}: \quad \lambda_{v,t} = \beta\lambda_{v,t+1}R_{t+1}. \tag{4.5.6}$$

Combining the first two optimality conditions reveals that hours are given by the simple static relationship

$$\varphi'(\ell_{v,t}) = w_t\theta_t \quad \Rightarrow \quad \ell_{v,t} = \ell_t. \tag{4.5.7}$$

Notice that because of our separability assumptions labor supply is just determined by the parametric specification of the disutility function of labor $\varphi(\cdot)$ and the wage rate. Hence, irrespective of age v every household will provide the same labor supply, i.e. $\ell_{v,t} = \ell_t$. Aggregate labor supply is therefore simply given as $L_t^S = \ell_t \theta_t N_t$. Using the same steps as in sections 4.1 and 4.2 implies that the aggregate consumption function is

$$Q_t = \Omega_t^{-1} \left(A_t + H_t \right), \tag{4.5.8}$$

where

$$H_t = w_t L_t^S - \varphi(\ell_t) N_t + \gamma_{t+1} \frac{H_{t+1}(N_t/N_{t+1})}{R_{t+1}}, \tag{4.5.9}$$

which uses the following aggregation results. Using the independence of optimal ℓ_t of age v and aggregating (4.5.3) reveals that

$$\sum_v C_{v,t} N_{v,t} = \sum_v Q_{v,t} N_{v,t} + \sum_v \varphi(\ell_t) N_{v,t} \Rightarrow C_t = Q_t + \varphi(\ell_t) N_t, \tag{4.5.10}$$

which can be used to back out aggregate 'pure' consumption C_t. The rest of the model is solved analogously to the exogenous labor supply Blanchard models presented in sections 4.1 and 4.2.

4.5.2 Extensive Margin

In this section we briefly sketch out the introduction of an additional margin, namely labor market participation. A thorough analysis and the implementation is left as an exercise. The participation decision occurs prior to the hours decision along the intensive margin. For tractability reasons we will make some simplifying assumptions. First, home production h^{tot} consists of two parts, a uniform fixed part[7] ω and a stochastic part h. Every household receives many, many i.i.d. shocks to the value of h during the period of one year. h is drawn from the distribution function $F(\cdot)$. This is obviously an unrealistic assumption but it will serve the purpose of thinking about participation in a probabilistic way and focus on average household behavior. The ex-ante probability of a household to draw a sufficiently small value of h such that participation in the labor market pays off is δ. In another interpretation δ is the share of time of a household spent in participation during a year.[8] Like hours supply participation is a static decision. A household

[7] The main purpose of this fixed part is to ease some comparative static exercises. It will be dropped again in the next section when we have explicit policy instruments working at the extensive margin.

[8] Next to the convenient feature that every household will have the same per period income which is necessary for aggregation this interpretation also circumvents a violation

will participate if the following is true

$$w_t \ell_{v,t} \theta_t - \varphi(\ell_{v,t}) > \omega + h_{v,t} \quad \Rightarrow \quad w_t \ell_{v,t} \theta_t - \varphi(\ell_{v,t}) - \omega = \underline{h}_{v,t}, \tag{4.5.11}$$

where $\underline{h}$ is the level of variable home production at which a household is indifferent. The probability of participating at a single instance is simply the cdf at the cut-off $\delta = F(\underline{h})$. Given the law of large numbers this is also the share of time spent in participation. Hence, yearly labor income per household is $y_{v,t} = \delta_{v,t} w_t \ell_{v,t} \theta_t$. Per period income enters the intertemporal budget constraint

$$\gamma_{t+1} A_{v,t+1} = R_{t+1} \left[A_{v,t} + y_{v,t} - C_{v,t} \right]. \tag{4.5.12}$$

Home production is integrated in the consumption bundle

$$Q_{v,t} = C_{v,t} - \delta_{v,t} \varphi\left(\ell_{v,t}\right) + (1 - \delta_{v,t}) \left[h^e_{v,t} + \omega \right], \tag{4.5.13}$$

where h^e is the conditional expectation, i.e. $h^e = F(\underline{h})^{-1} \int_{-\infty}^{\underline{h}} h \, dF(h)$. The two optimality and the envelope conditions using the usual definition of $\lambda_{v,t}$ are

$$C_{v,t}: \quad u'(Q_{v,t}) = \beta R_{t+1} \lambda_{v,t+1}, \tag{4.5.14}$$

$$\ell_{v,t}: \quad u'(Q_{v,t}) \delta_{v,t} \varphi'(\ell_{v,t}) = \beta R_{t+1} \lambda_{v,t+1} \delta_{v,t} w_t \theta_t \tag{4.5.15}$$

$$A_{v,t}: \quad \lambda_{v,t} = \beta \lambda_{v,t+1} R_{t+1}. \tag{4.5.16}$$

Clearly, the Euler equation is unaltered. Observe that we used the envelope theorem in the optimality condition for hours ℓ by setting $\partial \underline{h} / \partial \ell = 0$.[9] Combining the first two optimality conditions reveals that hours are again given by the simple static relationship

$$\varphi'(\ell_{v,t}) = w_t \theta_t \quad \Rightarrow \quad \ell_{v,t} = \ell_t, \tag{4.5.17}$$

i.e. independently of $\delta_{v,t}$, which again implies that every household irrespective of age v will supply the same number of hours (in case of participation) ℓ_t. This in turn means that also the cut-off $\underline{h}$ (see 4.5.11) and consequently

of expected utility.

[9] $\partial \underline{h} / \partial \ell = w_t \theta_t - \varphi'(\ell_{v,t})$ which is zero given the first order condition for ℓ (4.5.17).

the participation rate are independent of v, i.e. δ_t. Finally, this also implies homogeneous per period income y_t, which means that aggregation works as before.

4.5.3 Implementation

First we have to fix a functional form for the disutility function of hours, $\varphi(\cdot)$. We choose the following form

$$\varphi(\ell_t) = \varphi_0 \frac{\varepsilon_\ell}{1+\varepsilon_\ell} \left(\ell_t\right)^{\frac{1+\varepsilon_\ell}{\varepsilon_\ell}} - \varphi_1. \tag{4.5.18}$$

Computing the first derivative and inserting in the first order condition (4.5.17) reveals that

$$\ell_t = \left(\frac{w_t \theta_t}{\varphi_0}\right)^{\varepsilon_\ell}. \tag{4.5.19}$$

This implies that the elasticity of labor supply with respect to the wage rate or the productivity parameter is

$$\frac{\partial \ln \ell_t}{\partial \ln w_t} = \frac{\partial \ln \ell_t}{\partial \ln \theta_t} = \varepsilon_\ell, \tag{4.5.20}$$

i.e. a 1% increase in the wage rate leads to an ε_ℓ% increase in the individual hours choice. ε_ℓ therefore represents the micro-elasticity of labor supply because it governs the individual decision. In contrast if e.g. θ_t is increased by 1% for all workers the effect on average ℓ_t will not necessarily be equal to ε_ℓ. In this case we are looking for the macro-elasticity of labor supply which also include general equilibrium effects through changes in the wage rate. This elasticity is usually inferred from shocking the model and computing the change in average ℓ_t. For models with simple production like the ones presented before we can at least compute the long-run macro-elasticity, which in this case coincides with ε_ℓ. As a response to the productivity shock wages drop only in the beginning but recover to their initial value once the capital stock grows to its new steady state level. However, the short-run macro-elasticity will be smaller. The parameter φ_0 is a scaling parameter, i.e. to target a value of ℓ in the calibration. φ_1 is simply used to normalized the disutility costs $\varphi(\ell)$ in the calibration, e.g. to 0, which has no direct effect on the labor supply decision, but enters the consumption bundle Q.

For the participation margin it is convenient to assume that $F(\cdot)$ follows a Pareto distribution with scale parameter k and shape parameter κ. Then the non-participation rate is

$$(1-\delta) = \underline{h}^{-\kappa} k^{\kappa}, \tag{4.5.21}$$

where κ is used to capture the responsiveness of the margin, while k is used to target the initial level of participation. Under the condition that $\varphi(\ell)$ and ω are calibrated to be zero in the initial equilibrium κ can be interpreted directly as micro-elasticity, i.e. a 1% rise in the individual wage decreases the non-participation rate by κ% if hours are kept constant. Further we can explicitly solve for $h^e = F(\underline{h})^{-1} \int_{-\infty}^{\underline{h}} h\, dF(h)$. The pdf $f(h)$ is given as $\kappa h^{-\kappa-1} k^{\kappa}$. The partial expectation $\int_{k}^{\underline{h}} h f(h)\, dh = \frac{\kappa}{1-\kappa}\left[k^{\kappa}\underline{h}^{1-\kappa} - k\right]$. Therefore we can write the conditional expectation as

$$h^e = \frac{1}{(1-k^{-\kappa}\underline{h}^{\kappa})} \frac{\kappa}{(1-\kappa)} \left[k^{\kappa}\underline{h}^{1-\kappa} - k\right]. \tag{4.5.22}$$

The implementation of the intensive margin in the code is done in the code `Blanchard_intensive`. Incorporating the extensive margin is left as an exercise.

4.5.4 Exercises

Exercises

Ex. 6 — The extensive margin 1
Implement the extensive margin from section 4.5.2 in the code `Blanchard_intensive`. Target an initial participation rate of 0.7. An empirical paper tells you that the individual non-participation rate drops by 1.5% (not percentage points) for an individual increase of the wage rate by 1% if hours are kept constant. Leave the elasticity parameter of the hours supply unchanged for simplicity. Why could this be a bad idea?

Ex. 7 — The extensive margin 2
Starting from exercise 6, simulate a permanent introduction of welfare benefits

(payable in case of non-participation) amounting to 10% of the initial average wage income. Assume the increase to be uncompensated, i.e. financing happens outside of the model. Hint: Simply the fixed part of home production ω to mimic this policy. How and through which channels are the different margins affected?

4.6 Including Government

Summary

In this section we introduce a government into a Blanchard model in a small open economy setting with endogenous labor supply. Government can issue debt which is a perfect substitute for other household assets. Different tax instruments on all available margins are modeled. The readers are introduced to the concept of effective tax rates. We discuss the necessity of an explicit government debt rule when agents are perfectly forward looking. In addition we assume exogenous technological progress. The corresponding code is `Blanchard_government`.

In summary the following government instruments are considered in this section

- Income taxes from workers: τ^W
- Payroll taxes from firms: τ^F
- Lump-sum taxes/transfers from/to households: τ^l
- Profit taxes from firms: τ^{prof}
- Profit tax deductibility options for capital maintenance costs: ϕ_0^τ
- Consumption taxes: τ^C
- Unproductive government consumption: C^G
- Benefits for non-participating households: b

In this section we additionally introduce exogenous technological progress as explained in section 3.2 and detrend the model accordingly. The exogenous growth factor is $G = 1 + g$. The previous cases are nested for $g = 0$.

4.6.1 Households

Households make three decisions: consumption, labor supply along the intensive margin and labor supply along the extensive margin. For the participation decision people compare utility for both states and will participate whenever

$$(1-\tau_t^W)w_t\ell_{v,t}\theta_t - \tau_t^l - \varphi(\ell_{v,t})\cdot pc_t > h_{v,t}\cdot pc_t + b_t - \tau_t^l \quad \Rightarrow$$
$$\left[(1-\tau_t^W)w_t\ell_{v,t}\theta_t - b_t\right]/pc_t - \varphi(\ell_{v,t}) = \underline{h}_{v,t}, \qquad (4.6.1)$$

where $\underline{h}$ is the level of home production at which a household is indifferent. τ^W denotes a linear labor income tax for the worker, while b_t is a government sponsored welfare benefit in case of non-participation. Lump-sum taxes τ_t^l do not condition on the participation status, i.e. they have to be paid in any case and therefore play no role for participation (or any other labor supply decision). The disutility and the home production terms are multiplied by the price of consumption pc_t as they are defined in terms of consumption. See more on this below. Other than those three parameters modeling participation works as in section 4.5.2, i.e. $\delta\left(\underline{h}\right)$ can again be interpreted as share of time spent in participation during a year. Therefore, yearly labor income including welfare benefits is

$$y_{v,t} = \delta_{v,t}(1-\tau_t^W)w_t\ell_{v,t}\theta_t + (1-\delta_{v,t})b_t - \tau_t^l \qquad (4.6.2)$$

Assume an iso-elastic felicity function $u(x) = \frac{\sigma}{\sigma-1}x^{\frac{\sigma-1}{\sigma}}$ such that $u'(x) = x^{-\frac{1}{\sigma}}$, where σ is the intertemporal elasticity of substitution and $\rho = \frac{\sigma-1}{\sigma}$. The households optimization problem is

$$V(A_{v,t}) = \max_{C_{v,t},\ell_{v,t}} \; u(Q_{v,t}) + \beta\gamma_{t+1}G^{\rho}V(A_{v,t+1}) \quad \text{s.t.} \tag{4.6.3}$$

$$\gamma_{t+1}GA_{v,t+1} = R_{t+1}\left[A_{v,t} + y_{v,t} - pc_t C_{v,t}\right] \tag{4.6.4}$$

$$Q_{v,t} = C_{v,t} - \delta_{v,t}\varphi\left(\ell_{v,t}\right) + (1-\delta_{v,t})h^{e}_{v,t}, \tag{4.6.5}$$

and equation (4.6.2).

The price of consumption simply includes the tax rate on consumption, i.e. $pc_t \equiv 1 + \tau^{C}_{t}$. The optimality and envelope conditions using shadow price $\lambda_{v,t} \equiv V'(A_{v,t})$ are

$$C_{v,t}: \quad u'(Q_{v,t}) = \beta G^{\rho-1}R_{t+1}\lambda_{v,t+1}pc_t, \tag{4.6.6}$$

$$\ell_{v,t}: \quad u'(Q_{v,t})\delta_{v,t}\varphi'(\ell_{v,t}) = \beta G^{\rho-1}R_{t+1}\lambda_{v,t+1}\delta_{v,t}(1-\tau^{W}_{t})w_t\theta_t \tag{4.6.7}$$

$$A_{v,t}: \quad \lambda_{v,t} = \beta G^{\rho-1}\lambda_{v,t+1}R_{t+1}. \tag{4.6.8}$$

Combining the first two optimality conditions reveals that hours are again given by the simple static relationship

$$\varphi'(\ell_{v,t})\cdot pc_t = (1-\tau^{W}_{t})w_t\theta_t \quad \Rightarrow \quad \ell_{v,t} = \ell_t, \tag{4.6.9}$$

which establishes the independence of ℓ, $\underline{h}$ and δ of age v. From the first order conditions one can easily spot the different behavioral reactions to changes in our labor market instruments. An increase in welfare benefits b reduces participation δ, while there is no first order effect on the supplied hours. Labor tax τ^{W} discourages both, participation and hours supply. Combine the (4.6.6) and (4.6.8) to get the Euler equation

$$u'\left(Q_{v,t}\right) = \beta R_{t+1}\frac{pc_t}{pc_{t+1}}u'\left(Q_{v,t+1}\right). \tag{4.6.10}$$

Given the assumptions about the functional form of $u(\cdot)$ we can write the Euler equation as

$$GQ_{v,t+1} = \left(\beta R_{t+1}\frac{pc_t}{pc_{t+1}}\right)^{\sigma}Q_{v,t}. \tag{4.6.11}$$

Changes in the taxation of consumption influence the relative value of consumption today versus tomorrow. If pc_t/pc_{t+1} increases we will see a decrease of $Q_{v,t}/Q_{v,t+1}$. To derive the consumption function we take the same steps as in the previous sections. First let us define $\bar{y}_t = y_t - [\delta_t \varphi(\ell_t) - (1-\delta_t)h_t^e]\, pc_t$. Write the intertemporal budget constraint (4.6.4)as

$$\begin{aligned}
A_{v,t} &= [pc_t C_{v,t} - y_t] + \sum_{s=t+1}^{\infty} [pc_s C_{v,s} - y_s] \prod_{u=t+1}^{s} \frac{G\gamma_u}{R_u}, \quad \text{insert (4.6.5)} \\
A_{v,t} &= [pc_t Q_{v,t} - \bar{y}_t] + \sum_{s=t+1}^{\infty} [pc_s Q_{v,s} - \bar{y}_s] \prod_{u=t+1}^{s} \frac{G\gamma_u}{R_u} \quad \text{or} \\
\mathcal{W}_{v,t} &= A_{v,t} + H_{v,t}, \quad \text{where} \\
\mathcal{W}_{v,t} &\equiv pc_t Q_{v,t} + \sum_{s=t+1}^{\infty} pc_s Q_{v,s} \prod_{u=t+1}^{s} \frac{G\gamma_u}{R_u}, \quad H_{v,t} \equiv \bar{y}_t + \sum_{s=t+1}^{\infty} \bar{y}_s \prod_{u=t+1}^{s} \frac{G\gamma_u}{R_u}.
\end{aligned}$$

Insert the Euler equation consecutively to solve for the consumption function.

$$\begin{aligned}
\mathcal{W}_{v,t} &= pc_t Q_{v,t} + pc_t Q_{v,t} \left(\sum_{s=t+1}^{\infty} \beta^{(s-t)\sigma} \prod_{u=t+1}^{s} \left(R_u \frac{pc_{u-1}}{pc_u} \right)^{\sigma-1} \gamma_u \right) \\
&= pc_t Q_{v,t} \Omega_t, \qquad \text{where} \qquad (4.6.12) \\
\Omega_t &= 1 + \left(\sum_{s=t+1}^{\infty} \beta^{(s-t)\sigma} \prod_{u=t+1}^{s} \left(R_u \frac{pc_{u-1}}{pc_u} \right)^{\sigma-1} \gamma_u \right) \qquad (4.6.13)
\end{aligned}$$

Hence, the consumption function is

$$Q_{v,t} = (\Omega_t pc_t)^{-1} \mathcal{W}_{v,t} = (\Omega_t pc_t)^{-1} (A_{v,t} + H_{v,t}) \qquad (4.6.14)$$

where $A_{v,t}$, $H_{v,t}$ and the inverse marginal propensity Ω_t to consume are forward looking and can be recursively written as

$$\begin{aligned}
\Omega_t &= 1 + \beta^{\sigma} \left(R_{t+1} \frac{pc_t}{pc_{t+1}} \right)^{\sigma-1} \gamma_{t+1} \Omega_{t+1}, \qquad (4.6.15) \\
H_{v,t} &= \bar{y}_t + \gamma_{t+1} \frac{G H_{v,t+1}}{R_{t+1}}. \qquad (4.6.16)
\end{aligned}$$

Analogously to the sections before aggregation over v implies that

$$Q_t = (\Omega_t pc_t)^{-1} (A_t + H_t), \tag{4.6.17}$$
$$A_{t+1} = R_{t+1} [A_t + y_t N_t - pc_t C_t], \tag{4.6.18}$$
$$H_t = \bar{y}_t N_t + \gamma_{t+1} \frac{GH_{t+1}(N_t/N_{t+1})}{R_{t+1}}. \tag{4.6.19}$$

Although augmented[10] aggregate household income is now a more complex expression

$$\bar{y}_t N_t = (1-\tau_t^W) w_t L_t^S + (1-\delta_t) b_t N_t - \tau_t^l N_t - [\delta_t \varphi(\ell_t) - (1-\delta_t) h_t^e] pc_t N_t,$$

the characteristic of the system of difference equations was hardly altered. Total labor supply is defined as the supplied hours per participating person times number of persons, i.e. $L_t^S = \delta_t \ell_t \theta_t N_t$. Using the independence of optimal ℓ_t of age v and aggregating 4.6.5 reveals that

$$\sum_v C_{v,t} N_{v,t} = \sum_v Q_{v,t} N_{v,t} + \sum_v \delta_t \varphi(\ell_t) N_{v,t} - (1-\delta_t) h_t^e N_{v,t} \Rightarrow$$
$$C_t = Q_t + [\delta_t \varphi(\ell_t) - (1-\delta_t) h_t^e] N_t, \tag{4.6.20}$$

which can be used to back out aggregate 'pure' consumption C_t.

4.6.2 Production

The production function is assumed to be homogeneous of degree one[11], e.g. of a simple Cobb-Douglas form

$$Y_t = f(K_t, L_t^D) = A^0 (K_t)^\alpha \left(L_t^D\right)^{1-\alpha}, \tag{4.6.21}$$

where L^D is labor demand which in equilibrium obviously has to equal supply L^S. Capital adjustment through investment is paid out of per-period earnings as are capital adjustment costs J. Per-period after tax profits (i.e. dividends) are therefore

$$\chi_t = Y_t - (1+\tau_t^F) w_t L_t^D - I_t - J_t - T_t^F, \tag{4.6.22}$$

[10] 'Augmented' in the sense of adjusted for home production and disutility of labor.

[11] A^0 denotes total factor productivity and is not to be confused with assets at time t A_t.

where J_t is defined slightly differently to guarantee to again be zero in steady state

$$J(I_t, K_t) = \frac{1}{2}\psi K_t \left(\frac{I_t}{K_t} - (\delta^K + g)\right)^2. \tag{4.6.23}$$

τ^F are taxes related to the wage bill. T^F are profit taxes paid which we assume to take the following form

$$T_t^F = \tau_t^{prof} \left[Y_t - (1+\tau_t^F)w_t L_t^D - \phi_0^\tau \left(J_t + \delta^K K_t\right)\right]. \tag{4.6.24}$$

For the taxable profit firms can deduct wage costs from revenues. The tax system parameter $\phi_0^\tau \in \{0,1\}$ controls whether a deduction of capital replacement investment from the tax base is allowed or not. Note that because investments are made from retained earnings the 'true' capital costs will always exceed the deductible part as the opportunity costs of internal financing is assumed to be non-deductible.[12] Inserting for T^F gives a single expression for dividends χ

$$\chi_t = (1-\tau_t^{prof})\left[Y_t - (1+\tau_t^F)w_t L_t^D\right] - I_t - (1-\phi_0^\tau \tau_t^{prof})J_t + \phi_0^\tau \tau^{prof}\delta^K K_t. \tag{4.6.25}$$

The value of the firm is the discounted stream of per-period profits, i.e.

$$V_t = \chi_t + \frac{GV_{t+1}}{R_{t+1}}, \tag{4.6.26}$$

and the law of motion for capital is

$$GK_{t+1} = (1-\delta^K)K_t + I_t. \tag{4.6.27}$$

The firm solves the following problem

$$V(K_t) = \max_{I_t, L_t^D} \ \chi_t + \frac{GV(K_{t+1})}{R_{t+1}}. \tag{4.6.28}$$

[12]As we will see later, this implies that the tax distortion cannot be undone using the assumed method of deductibility.

Define the marginal benefit of an increase in capital $q_t \equiv V'(K_t)$. The optimality and envelope conditions are

$$\begin{aligned}
I_t: &\quad q_{t+1} = R_{t+1}(1 + (1 - \phi_0^\tau \tau_t^{prof}) J_{I_t}) && (4.6.29)\\
L_t^D: &\quad Y_{L_t^D} = (1 + \tau_t^F) w_t && (4.6.30)\\
K_t: &\quad q_t = (1 - \tau_t^{prof}) Y_{K_t} - (1 - \phi_0^\tau \tau_t^{prof}) J_{K_t} + \\
&\quad \phi_0^\tau \tau_t^{prof} \delta^K + \frac{q_{t+1}}{R_{t+1}}(1 - \delta^K). && (4.6.31)
\end{aligned}$$

The pay-roll tax rate τ_t^F clearly raises effect labor costs and therefore reduces labor demand. To see how the other tax instruments work, let us for the moment focus on the steady state where $J = J_I = J_K = 0$. Combine (4.6.29) with (4.6.31) and rearrange to get

$$Y_K = \frac{r + \delta^K(1 - \phi_0^\tau \tau^{prof})}{1 - \tau^{prof}}, \tag{4.6.32}$$

where the right-hand side reflects effective user costs of capital. Observe how in absence of any tax instrument this would simply be $r + \delta^K$. The profit tax rate τ^{prof} clearly increases user cost of capital and therefore reduces capital usage. In case that $\phi_0^\tau = 1$, the tax system is not completely neutral as user cost of capital is reduced to $r/(1 - \tau^{prof}) + \delta^K$ which however is lower than $(r + \delta^K)/(1 - \tau^{prof})$ if $\phi_0^\tau = 0$.

We briefly establish Hayashi (1982)'s theorem 3.1.1, which connects the firm value V and the capital stock K is also true in the case government intervention in production of the forms described above.

Proof. Take the envelope condition for K_t (4.6.31) and multiply both sides by K_t and expand the right hand side by $\frac{q_{t+1}}{R_{t+1}} I_t$.

$$
\begin{aligned}
q_t K_t &= (1-\tau_t^{prof}) Y_{K_t} K_t - (1-\phi_0^\tau \tau^{prof}) J_{K_t} K_t + \phi_0^\tau \tau_t^{prof} \delta^K K_t \\
&\quad + \frac{q_{t+1}}{R_{t+1}} \left[(1-\delta^K) K_t + I_t \right] - \frac{q_{t+1}}{R_{t+1}} I_t, \\
q_t K_t &= (1-\tau_t^{prof}) \left[Y_{K_t} K_t + Y_{L_t^D} L_t^D - Y_{L_t^D} L_t^D \right] + \phi_0^\tau \tau_t^{prof} \delta^K K_t \\
&\quad - (1-\phi_0^\tau \tau_t^{prof}) J_{K_t} K_t - \frac{q_{t+1}}{R_{t+1}} I_t + \frac{q_{t+1}}{R_{t+1}} G K_{t+1}, \\
q_t K_t &= (1-\tau_t^{prof}) \left[Y_t - (1+\tau_t^F) w_t L_t^D \right] + \phi_0^\tau \tau_t^{prof} \delta^K K_t \\
&\quad - (1-\phi_0^\tau \tau_t^{prof}) J_{K_t} K_t - (1 + (1-\phi_0^\tau \tau_t^{prof}) J_{I_t}) I_t + \frac{q_{t+1}}{R_{t+1}} G K_{t+1}, \\
q_t K_t &= \chi_t + \frac{q_{t+1}}{R_{t+1}} G K_{t+1}.
\end{aligned}
$$

From the first to the second equation we used the law of motion (4.6.27) and expanded by the term $(1-\tau_t^{prof}) Y_{L_t^D} L_t^D$. From the second to the third equation we used Euler's theorem and the linear homogeneity of the production function. We further use the optimality conditions for labor demand and investment. From the third to the fourth line we use Euler's theorem and the linear homogeneity of the capital adjustment cost function and afterwards use the definition of per-period profits χ. Solving forward yields Hayashi (1982)'s result

$$
q_t K_t = \sum_{s=t}^{\infty} \chi_s \prod_{u=t+1}^{s} \frac{G}{R_u} = V_t \tag{4.6.33}
$$

■

Using the law of motion for capital (4.6.27) and Hayashi's theorem evaluated at the optimality condition for investment, i.e. $q_{t+1} = R_{t+1}(1 + (1 - \phi_0^\tau \tau_t^{prof}) J_{I_t})$ gives an implicit relation for investment:

$$
I_t = \frac{G V_{t+1}}{R_{t+1}(1 + (1-\phi_0^\tau \tau_t^{prof}) J_{I_t})} - (1-\delta^K) K_t. \tag{4.6.34}
$$

Using our assumed functional form for J_t we can solve explicitly for I_t as the solution to the following quadratic equation:

$$\left[\frac{\widetilde{\psi}}{K_t}\right] \cdot I_t^2 + \left[\left[1 - \widetilde{\psi}(\delta^K + g)\right] + (1 - \delta^K)\widetilde{\psi}\right] \cdot I_t$$
$$+ \left[(1 - \delta^K)\left[1 - \widetilde{\psi}(\delta^K + g)\right] K_t - \frac{GV_{t+1}}{R_{t+1}}\right] = 0, \qquad (4.6.35)$$

where $\widetilde{\psi} = \psi(1 - \phi_0^\tau \tau_t^{prof})$.

4.6.3 Government

The government sector is characterized by the following. First, it raises revenue according to the tax bases described above. Second, the government consumes the homogeneous good. In contrast to households this consumption is not micro-founded but exogenously given by $C_t^G \geq 0$. We assume that this consumption is not taxed.[13] Government consumption (already by its name) is assumed to be purely consumptive, i.e. this type of expenditure is not meant to invest in some form of public capital stock, e.g. infrastructure, education, etc. which would influence a countries productivity. Third, government provides social security, in this simple model, by granting welfare benefits to non-participants. Fourth, the government can issue debt in form of government bonds which by assumption are perfect substitutes for the other assets (firm shares and foreign assets). Per period government revenue is given by

$$Rev_t = T_t^F + \left(\tau_t^F L_t^D + \tau_t^W L_t^S\right) w_t + \tau_t^l N_t + \tau_t^C C_t, \qquad (4.6.36)$$

while expenditure excluding interest payments for the government debt are

$$Exp_t = C_t^G + (1 - \delta_t) b_t N_t. \qquad (4.6.37)$$

For some applications it might be convenient to define $C_t^G = c_t^G N_t$, e.g. to capture increasing costs of public consumption in case of population growth.

[13] Economically this would not make a difference because the government would simply put money from one to another pocket. However, to match empirical value added tax (VAT) shares it might make sense to relax this assumption as in reality parts of government consumption are indeed subject to VAT.

The primary balance is consequently defined as

$$PB_t = Rev_t - Exp_t, \tag{4.6.38}$$

which changes the stock of public debt according to

$$GD^G_{t+1} = R_{t+1}\left(D^G_t - PB_t\right). \tag{4.6.39}$$

One has to set at least one policy rule for the government, namely, one that targets long-run debt at a level[14] that is bounded from above as well as below

$$\exists B : \lim_{t\to\infty} D^G_t = B \text{ and } |B| < \infty, \tag{4.6.40}$$

otherwise no final steady state would exist. The policy rule has to involve at least one government instrument as well as a feasible path of the evolution of D^G. Note that for example big immediate downward jumps, e.g. $D^G_t - GD^G_{t+1} \gg C^G_t$ might prove to be unfeasible as this jump might imply such a severe tax increase which simply cannot be achieved due to Laffer curve effects. This is obviously not true for lump-sum taxes, however, next to being an unrealistic instrument mostly introduced for academic purpose there are also reasonable limits to this instrument by assuming that household assets have to be bounded from below at some level. An example of a feasible policy rule starting from D^G_0 would be to adjust employees' wage taxes τ^W_t every period in order to have $D^G_t = D^G_0,\ \forall t > 0$. Another would be to fix all instruments for 20 years and let D^G change freely and then use changes in government consumption to fix government debt at its current level, i.e. $D^G_t = D^G_{t=20},\ \forall t > 20$.

4.6.4 Intermezzo: Effective Tax Rates

A single decision margin is often influenced by many different tax instruments. It can be very insightful to capture all the relevant instruments in a single - decision margin related - effective tax rate or tax wedge. We denote effective tax rates using the hat notation and start with the

[14]Note that in a model with exogenous growth one has to target a bounded level of government debt in detrended terms.

decision margin for hours supplied. Observe that the relevant first order condition (4.6.9) can be rearranged to depend on a single instrument,

$$\varphi'(\ell_t) = (1 - \hat{\tau}_t^W) w_t \theta_t, \quad \hat{\tau}_t^W = \frac{\tau_t^C + \tau_t^W}{1 + \tau_t^C} \tag{4.6.41}$$

where $\hat{\tau}_t^W$ is the effective tax rate for supplying hours. The usefulness becomes more apparent for the extensive margin. We rearrange (4.6.1) to get

$$(1 - \hat{\tau}_t^\delta) w_t \ell_t \theta_t - \varphi(\ell_t) = \underline{h}_t, \quad \hat{\tau}_t^\delta = \frac{\tau_t^C + \tau_t^W + b_t/(w_t \ell_t \theta_t)}{1 + \tau_t^C}. \tag{4.6.42}$$

The effective tax rate $\hat{\tau}_t^\delta$ - in this case also referred to as the 'participation tax rate' - is increasing in the wage tax rate as well as in the replacement rate $b_t/(w_t \ell_t \theta_t)$. Hence, the perceived taxation can be relatively high if governments grant generous benefits. The formulas above suggest that the effective tax for participation is always higher than for supplied hours. This is not necessarily true in general. The presented model assumes linear tax schedules, which implies that marginal and average tax rates coincide. For non-linear tax schedules this is not the case and this is important because the intensive margin decision is influenced by the marginal tax rate, while the extensive margin depends only on the average tax rate. One can easily apply the idea of effective taxation to other margins, e.g. the capital usage decision of the firm. We rearrange (4.6.32) and define the effective profit tax in steady state as

$$(1 - \hat{\tau}^{prof}) Y_K = r + \delta^K, \quad \hat{\tau}^{prof} = 1 - \frac{(1 - \tau^{prof})(r + \delta^K)}{r + \delta^K (1 - \phi_0^\tau \tau^{prof})}. \tag{4.6.43}$$

4.6.5 Temporary Equilibrium

Before stepping through the process of computing the temporary equilibrium we have to update the market clearing conditions for assets, labor, goods and the government budget.

$$A_t = V_t + D_t^F + D_t^G \tag{4.6.44}$$
$$L_t^S = L_t^D \tag{4.6.45}$$
$$Y_t = C_t + I_t + J_t + C_t^G + TB_t \tag{4.6.46}$$
$$PB_t = \text{ target according to the chosen policy rule.} \tag{4.6.47}$$

The changed goods market condition also implies an adjustment for the trade balance

$$GD_{t+1}^F = R_{t+1}\left[D_t^F + TB_t\right], \quad \text{where} \quad TB_t \equiv Y_t - C_t - I_t - J_t - C_t^G, \tag{4.6.48}$$

In this section we present the computation of the temporary equilibrium at t in greater detail in an algorithmic form in order to illustrate better how exactly the implementation on the computer works. Some of steps could in principle obviously be swapped. We start with the following information. $r_t \ \ \forall t$, K_t, D_t^F, D_t^G, N_t and all government instruments except for one are either exogenously given or predetermined. We take guesses for the foresight variables[15] $\Omega_{t+1}, V_{t+1}, H_{t+1}$ as given. The presented steps will lead to a system of two unknowns (w_t and one policy instrument) and two equations.

1. Compute N_{t+1} from (4.3.2).[16]
2. Compute L_t^D from (4.6.30).
3. Compute Y_t from (4.6.21).
4. Compute ℓ_t from (4.6.9).
5. Compute $\underline{h}_t$, δ_t and h_t^e from (4.6.1), etc.
6. Compute L_t^S from its definition.
7. Compute $\bar{y}$ from its definition.
8. Compute I_t from (4.6.35).

[15] If τ^C is the free policy instrument we also have to make a guess for pc_{t+1}.
[16] As equilibrium is fully recursive w.r.t. demography one can compute the demographic development at any point in the algorithm completely detached from the economic problem.

9. Compute χ_t from (4.6.25).
10. Compute V_t from (4.6.26).
11. Compute K_{t+1} from (4.6.27).
12. Compute Ω_t from (4.6.15).
13. Compute H_t from (4.6.19).
14. Compute Q_t from (4.6.17) and (4.6.44).
15. Compute C_t from (4.6.20).
16. Compute A_{t+1} from (4.6.18).
17. Compute the two unknowns by solving the system (4.6.45) and (4.6.47).
18. Compute D^F_{t+1} from (4.6.48).
19. Compute D^G_{t+1} from (4.6.39).

4.6.6 Walras' Law

Define the following excess demands

$$\begin{aligned} \text{assets}: &\quad \zeta^A_t = V_t + D^F_t + D^G_t - A_t && (4.6.49)\\ \text{labor}: &\quad \zeta^L_t = L^D_t - L^S_t && (4.6.50)\\ \text{goods}: &\quad \zeta^Y_t = C_t + I_t + J_t + C^G_t + TB_t - Y_t && (4.6.51)\\ \text{government}: &\quad \zeta^G_t = Rev_t - Exp_t - PB_t && (4.6.52) \end{aligned}$$

Rewrite (4.6.49) by inserting for V_t using (4.6.26) and eliminate χ_t by using (4.6.22) to get:

$$A_t = -\zeta^A_t + D^F_t + D^G_t + Y_t - (1+\tau^F_t)w_t L^D_t - I_t - J_t - T^F_t + \frac{GV_{t+1}}{R_{t+1}}. \quad (4.6.53)$$

Insert this expression in the aggregate intertemporal budget constraint (4.6.18) to get

$$G\frac{A_{t+1} - V_{t+1}}{R_{t+1}} = -\zeta^A_t + D^F_t + D^G_t + Y_t - (1+\tau^F_t)w_t L^D_t - I_t - J_t - T^F_t + y_t N_t - pc_t C_t.$$

Insert for $y_t N_t$ and rearrange terms to arrive at

$$G\frac{A_{t+1}-V_{t+1}}{R_{t+1}} = -\zeta_t^A + D_t^F + D_t^G - w_t\left(L_t^D - L_t^S\right) + Y_t - I_t - J_t - C_t \\ + (1-\delta_t)b_t N_t - T_t^F - w_t\left(\tau_t^F L_t^D + \tau_t^W L_t^S\right) - \tau_t^l N_t - \tau_t^C C_t.$$

Insert (4.6.50) and (4.6.51) to get

$$G\frac{A_{t+1}-V_{t+1}}{R_{t+1}} = -\zeta_t^A + D_t^F + D_t^G - w_t\zeta_t^L - \zeta_t^Y + TB_t + C_t^G \\ + (1-\delta_t)b_t N_t - T_t^F - w_t\left(\tau_t^F L_t^D + \tau_t^W L_t^S\right) - \tau_t^l N_t - \tau_t^C C_t.$$

Now use the definitions of Rev_t and Exp_t from (4.6.36) and (4.6.37) and insert 4.6.52. Afterward insert (4.6.49) at $t+1$.

$$G\frac{D_{t+1}^F + D_{t+1}^G - \zeta_{t+1}^A}{R_{t+1}} = -\zeta_t^A + D_t^F + D_t^G - w_t\zeta_t^L - \zeta_t^Y + TB_t - \zeta_t^G - PB_t.$$

Insert the laws of motion for foreign assets (4.2.12) and government debt (4.6.39) to arrive at Walras' Law

$$\zeta_t^Y + w_t\zeta_t^L + \zeta_t^A + \zeta_t^G - \frac{G\zeta_{t+1}^A}{R_{t+1}} = 0. \tag{4.6.54}$$

Consequently, the steady state version of Walras' Law is

$$\zeta^Y + w\zeta^L + \zeta^G + \frac{r-g}{R}\zeta^A = 0. \tag{4.6.55}$$

4.6.7 Steady State

Like for the temporary equilibrium section we use this section to explain the steps of computing the steady state in more detail. Again some of steps could in principle be swapped. We start with the following information in steady state. r, all other deep parameters and all government instruments except for one which is used to balance government budget are known. The presented steps will lead to a system of three unknowns (w, D^F and one policy instrument) and three equations. The following equation references always refer to their steady-state versions.

1. Compute N from (4.3.2).

2. Compute ℓ from (4.6.9).
3. Compute $\underline{h}$, δ and h^e from (4.6.1), etc.
4. Compute L^S from its definition.
5. Compute K from (4.6.32) in anticipation of labor market equilibrium, set $L^D = L^S$.
6. Compute Y from (4.6.21).
7. Compute L^D from (4.6.30)
8. Compute $\bar{y}$ from its definition.
9. Compute I from (4.6.27)
10. Compute χ from (4.6.25).
11. Compute V from (4.6.26).
12. Compute Ω from (4.6.15).
13. Compute H from (4.6.19).
14. Compute Q from (4.6.17) and (4.6.44).
15. Compute C from (4.6.20).
16. Compute A from (4.6.18).
17. Compute the three unknowns by solving the system (4.6.45) to (4.6.47).
18. Compute D^G from (4.6.39).

4.6.8 Implementation

The model is implemented in the code `Blanchard_government`. The different possiblilities of balancing government budget are integrated using a convenient switch. As default debt rule the code uses a rule where the detrended debt level is constant every period. More sophisticated rules are left as an exercise. Starting now all codes use explicit checks for Walras' Law and steady state as well as temporary equilibrium. Consistently checking Walras' Law is a helpful routine in order to avoid bugs in the model.

4.7 Exercises

Exercises

Ex. 8 — Fiscal Devaluation

Simulate a fiscal devaluation, i.e. reduce workers' labor taxes by 2 %-points financed through an (ex-post) debt-neutral raise in consumption taxes. Discuss labor supply and consumption behavior. What are the effects on the participation tax rate and the effective tax rate for hours supply? What happens to the trade balance? How robust is the effect on the latter to changes in the discount rate?

Ex. 9 — Debt-rule and Ricardian Equivalence

Based on the code `Blanchard_government`, simulate a decrease of lump-sum taxes by 2%. Use income taxes as the endogenous tax instrument. First, do the simulation under the default debt-rule that detrended debt should stay constant every period. Second, change the code such that there is no debt rule for the first 10 years, i.e. debt accumulates freely. After that the debt level is consolidated to its original level (linearly over 10 years). Third, set the length of the no-debt rule phase and the subsequent consolidation phase to 100 instead of 10 years each. Discuss the results. Compare especially the different effects on consumption. How does this relate to Ricardian Equivalence. Implementation hint: For changing the debt rule 'hard-code' the changes directly in `TE.m`.

4.8 Intermezzo: Progressive Taxation

Progressive taxation has the following two characteristics: First, the average tax rate increases in the tax base (B). Second, the marginal tax rate exceeds the average tax rate for every value of $B > 0$. See intermezzo 4.9 for more details. We will discuss three forms of modeling a progressive tax structure.

Proportional tax with a demogrant

The easiest possibility to introduce progressivity is to use a proportional marginal tax rate τ and a demogrant, i.e. a lump sum transfer independent of the tax base. In this case

$$\begin{aligned} \text{tax payment:} \quad & \tau B - D \\ \text{marginal tax rate:} \quad & \tau \\ \text{average tax rate:} \quad & \tau - D/B \end{aligned}$$

This can be a sufficient representation, especially in a more stylized model where only the principle distinction between marginal and average tax rate is of importance. The former is the relevant tax rate for decisions along the intensive margin while the latter is the determinant tax rate for decisions at the extensive margin.

Tax brackets with increasing marginal tax rates

A more realistic approach is to model tax brackets as they are used in many countries explicitly. In contrast to the previous modeling approach it enables us to take bracket creep into account, because marginal taxes rates rise with the tax base. First, assume that there are m tax brackets with marginal tax rates $\tau_1 < \tau_2 < \ldots < \tau_m$. The characteristic of the tax structure is that different marginal tax rates are applied to different tax brackets

$$\begin{aligned} \text{from } B_1 \text{ to } B_2 \ & : \tau_1 \\ \text{from } B_2 \text{ to } B_3 \ & : \tau_2 \\ & \vdots \\ \text{from } B_m \ & : \tau_m \end{aligned}$$

Define a tax payer of type i as $i = \max\{j : B_j \leq B\}$. Consequently, there are m types of tax payers. Define $\Omega_i = \sum_{j=1}^{i-1} [B_{j+1} - B_j] \tau_j$ with $\Omega_1 = 0$. We can now describe tax payment and the tax rates for tax payer i.

$$\begin{aligned} \text{tax payment:} \quad & \Omega_i + [B - B_i] \tau_i \\ \text{marginal tax rate:} \quad & \tau_i \\ \text{average tax rate:} \quad & \tau_i + [\Omega_i - B_i] / B \end{aligned}$$

Observe that the average tax rates is indeed increasing in B as $\Omega_i < B_i$ because of the assumption of increasing marginal tax rates. Further, note that the qualitative characteristics are very similar to the simpler version from above. The important difference now is that one can capture situation in which tax payer i becomes of type $j \neq i$.

Smoothly increasing marginal tax rates

A few tax systems in fact feature smoothly increasing marginal tax rates (e.g. the German income tax). However, the systems of many more countries might be characterized by smoothly increasing effective marginal tax rates if certain transfer systems (e.g. in-work-benefits that are smoothly phased out) are included in the representation, e.g. if one wants to capture the whole tax-benefit instead of only the tax system. In addition in many cases it might be convenient to approximate tax schedules that are subject to complicated rules with discrete jumps using smooth functions. Assume we can observe the average tax payment $T(B)$ for different tax bases B. Tax payment can then be approximated using an n-order polynomial from which approximate values for average and marginal tax rates can be backed out.

$$\begin{aligned} \text{tax payment:} \quad & \sum_{j=0}^{n} \hat{\beta}_j (B)^j \\ \text{marginal tax rate:} \quad & \sum_{j=1}^{n} \hat{\beta}_j (B)^{j-1} j \\ \text{average tax rate:} \quad & \sum_{j=0}^{n} \hat{\beta}_j (B)^{j-1} \end{aligned}$$

One has to make sure that the approximation is particularly good where (a) the mass of tax payers is and (b) where the mass of the tax base is concentrated. Given the low heterogeneity of household incomes that is used in the presented models we refrained from using one of the presented progressive taxation formulations.

4.9 Intermezzo: Interpreting Average and Marginal Tax Rates

The distinction between **average and marginal taxation** is important for two reasons. First, as will be discussed in section 6.4.2 they work at different margins. The average tax rate is an important determinant for decisions along the extensive margin while the marginal tax rate affects decisions along the intensive margin. Second, the difference between average and marginal tax rate is an important indicator for the progressivity of the tax system. Let B be the tax base and $T(B)$ the tax liability. In intermezzo 4.8 a progressive (regressive) tax system was defined by a an average tax rate increasing (decreasing) in the tax base. The second characteristic is a direct consequence of this definition as $d(T(B)/B)/dB = \left[T'(B) - T(B)/B\right]/B$. We discuss two indicators of measuring local progressivity (see Musgrave and Thin (1948)).

Coefficient of Liability Progression
The coefficient of liability progression (ε) is defined as the elasticity of tax liability w.r.t. the tax base, i.e.

$$\varepsilon = \frac{dT(B)}{dB}\frac{B}{T(B)} = \frac{T'(B)}{T(B)/B}. \tag{4.9.1}$$

Hence, ε is equal to the ratio of marginal and average tax rate and is bigger (smaller) than 1 if the system is progressive (regressive). A weighted average of all individual ε for all different B is an important macro indicator for tax revenue forecasting.

Coefficient of Residual Income Progression
The coefficient of residual income progression (η) is defined as the elasticity of net income w.r.t. the tax base, i.e.

$$\eta = \frac{dB - T(B)}{dB}\frac{B}{B - T(B)} = \frac{1 - T'(B)}{1 - T(B)/B}. \tag{4.9.2}$$

Chapter 5

OLG - The Gertler Model and Probabilistic Aging

Summary

In this section we extend the Blanchard model in a small open economy setting with endogenous labor supply and a government sector to a two-age-class model in the spirit of Gertler (1999). The two-age groups are interpreted as workers and retirees which differ in labor supply and income streams but can still separately be aggregated similar to the Blanchard model. This feature allows modeling of a pension system. The section then further discusses the generalization of the Gertler model to A age-classes known as 'Probabilistic Aging' as proposed in Grafenhofer et al. (2007). The corresponding code is `Gertler`.

5.1 The Gertler Model

5.1.1 Description of the Economy

The economy is almost identical to the small open economy Blanchard model with endogenous labor supply and existence of a government. The two age-classes differ in the following way. Young workers supply labor and in case of participation pay contributions that not only finance non-participation benefits for the young and government consumption but also old-age pension benefits.

5.1.2 Demography

Let us use a as superscript index for the age-class which can either be w (worker) or r (retiree). Both face a mortality risk by dying with probability $1-\gamma^a$ every period, where we assume that $\gamma^w > \gamma^r$.[1] In addition workers face an 'aging risk' capture by the feature that with probability $1-\omega$ they will age into the next age class and retiree. Hence, in absence of a mortality shock for the young workers will on average stay $1/(1-\omega)$ periods in the working age class. Similarly to the Blanchard model we have to keep track of the demographic characteristics of a household. But instead of storing the time of birth we also have to keep track of the time of retirement. We assume that all this relevant information is stored in a biography α. If a household retirees the biography is updated from α to α', with the consequence that any variable X_α does not necessarily coincide with $X_{\alpha'}$. The optimization problem is later solved for a representative household with some biography α. The evolution of the mass of persons of a specific biography α is therefore

$$\begin{aligned}
&\text{death:} && N^{\dagger}_{\alpha,t+1} = N^a_{\alpha,t} \cdot (1-\gamma^a_{t+1}),\ \forall a,\\
&\text{no aging (worker):} && N^w_{\alpha,t+1} = N^w_{\alpha,t} \cdot \gamma^w_{t+1}\omega,\\
&\text{no aging (retiree):} && N^r_{\alpha,t+1} = N^r_{\alpha,t} \cdot \gamma^r_{t+1},\\
&\text{aging:} && N^r_{\alpha',t+1} = N^w_{\alpha,t} \cdot \gamma^w_{t+1}(1-\omega).
\end{aligned}$$

The mass of households within an age-class simply follows from aggregating over all possible biographies, i.e. $N^a_t = \sum_\alpha N^a_{\alpha,t}$. Using the law of large numbers this implies the following evolution w.r.t. age-classes[2]

$$N^r_{t+1} = \gamma^r_{t+1} N^r_t + \gamma^w (1-\omega) N^w_t, \tag{5.1.1}$$

$$N^w_{t+1} = \gamma^w_{t+1} \omega N^w_t + NB_{t+1}. \tag{5.1.2}$$

[1]The original Gertler (1999) model assumes $\gamma^w = 1$, i.e. no mortality risk for the workers. The presented specification nests this special case.

[2]See Grafenhofer et al. (2007) for thorough proof.

5.1.3 Households

The income streams for a household with biography α is

$$y^a_{\alpha,t} = \begin{cases} \delta_{\alpha,t}(1-\tau^W_t)w_t\ell_{\alpha,t}\theta_t + (1-\delta_{\alpha,t})b_t - \tau^l_t & \text{if } a = w \\ P_t - \tau^l_t & \text{if } a = r \end{cases} \tag{5.1.3}$$

Further, define $\bar{y}_{\alpha,t} = y^w_{\alpha,t} - \left[\delta_{\alpha,t}\varphi\left(\ell_{\alpha,t}\right) - (1-\delta_{\alpha,t})h^e_{\alpha,t}\right]pc_t$. Observe that income could in principle depend on the individual biographies because the choices of hours ℓ and participation δ might. We will later show that in equilibrium those two choices, given our preference assumption will not differ for different biographies, hence, $y^a_{\alpha,t} = y^a_t \ \forall\alpha$. Similarly to before using a reverse life insurance[3] the intertemporal budget constraint is

$$\gamma^a_{t+1}GA^a_{\alpha,t+1} = R_{t+1}\left[A^a_{\alpha,t} + y^a_{\alpha,t} - pc_tC^a_{\alpha,t}\right], \quad \text{with} \quad A^w_{\alpha,t+1} = A^r_{\alpha',t+1}. \tag{5.1.4}$$

Observe the last part stating that the assets stock is unchanged during the process of retiring. We explain the two household problems for retirees and workers separately.

Retirees

The problem of retirees is identical to that of the Blanchard model with exogenous labor supply (as retirees do not work anymore). Preferences are given by a special non-expected utility CES form as proposed by Farmer (1990). It implies that households are risk-neutral while still an arbitrary intertemporal elasticity of substitution σ with $\rho = \frac{\sigma-1}{\sigma}$ can be used. This deviation from standard preferences does not impact retirees but it implies that workers will be risk-neutral concerning the aging risk. The households optimization for the retirees is

$$V^r_{\alpha,t} = \max_{C_{\alpha,t}} \ \left[(C^r_{\alpha,t})^\rho + \beta\gamma^r_{t+1}\left(GV^r_{\alpha,t+1}\right)^\rho\right]^{1/\rho} \quad \text{s.t. } (5.1.4), \tag{5.1.5}$$

where $V^r_{\alpha,t}$ is short for $V(A^r_{\alpha,t})$. The optimality and envelope conditions using

[3]There are in fact two reverse life insurances: one for the workers and one for the retirees.

shadow price $\lambda^r_{\alpha,t} \equiv V'(A^r_{\alpha,t}) \cdot (V^r_{\alpha,t})^{\rho-1}$ are

$$C^r_{\alpha,t}: \quad (C^r_{\alpha,t})^{\rho-1} = \beta R_{t+1} G^{\rho-1} \lambda^r_{\alpha,t+1} pc_t, \tag{5.1.6}$$

$$A^r_{\alpha,t}: \quad \lambda^r_{\alpha,t} = \beta G^{\rho-1} \lambda^r_{\alpha,t+1} R_{t+1}. \tag{5.1.7}$$

Combining (5.1.6) and (5.1.7) results in exactly the same Euler equation as before in the Blanchard model

$$GC^r_{\alpha,t+1} = \left(\beta R_{t+1} \frac{pc_t}{pc_{t+1}}\right)^\sigma C^r_{\alpha,t}. \tag{5.1.8}$$

As the budget constraint is also the same as before we conjecture that the consumption function is again given by

$$C^r_{\alpha,t} = (\Omega^r_t)^{-1} (A^r_{\alpha,t} + H^r_{\alpha,t}) \tag{5.1.9}$$

where human wealth $H^r_{\alpha,t}$ and the inverse marginal propensity Ω^r_t to consume are forward looking and can be recursively written as

$$\Omega^r_t = 1 + \beta^\sigma \left(R_{t+1} \frac{pc_t}{pc_{t+1}}\right)^{\sigma-1} \gamma^r_{t+1} \Omega^r_{t+1}, \tag{5.1.10}$$

$$H^r_{\alpha,t} = y^r_t + \gamma^r_{t+1} \frac{GH^r_{\alpha,t+1}}{R_{t+1}}. \tag{5.1.11}$$

Workers

The solution to the household problem of the workers is more difficult for two reasons. First, they in addition also have to optimize the labor supply. This was already covered in section 4.6.1 and works analogously again. Hence, this will not be discussed in further detail here. The important thing is that both labor supply decisions again are independent of the biographies α - a result that carries over from the Blanchard model. The problem of the worker looks as follows

$$V^w_{\alpha,t} = \max_{C^w_{\alpha,t}, l^w_{\alpha,t}} \left[(C^w_{\alpha,t})^\rho + \beta \gamma^w_{t+1} \left(G\bar{V}^w_{\alpha,t+1}\right)^\rho\right]^{1/\rho} \tag{5.1.12}$$

$$\text{s.t. (5.1.3) and (5.1.4),}$$

$$Q_{\alpha,t} = C^w_{\alpha,t} - \delta_{\alpha,t}\varphi(\ell_{\alpha,t}) + (1-\delta_{\alpha,t}) h^e_{\alpha,t} \tag{5.1.13}$$

$$\bar{V}^w_{\alpha,t+1} = \omega V^w_{\alpha,t+1} + (1-\omega) V^r_{\alpha',t+1} \tag{5.1.14}$$

Define the following shadow prices

$$\lambda^w_{\alpha,t} = \frac{dV^w_{\alpha,t}}{dA^w_{\alpha,t}} \left(V^w_{\alpha,t}\right)^{\rho-1}, \ \bar{\lambda}^w_{\alpha,t+1} = \left[\omega \frac{dV^w_{\alpha,t+1}}{dA^w_{\alpha,t+1}} + (1-\omega)\frac{dV^r_{\alpha',t+1}}{dA^r_{\alpha',t+1}}\right] \left(\bar{V}^w_{\alpha,t+1}\right)^{\rho-1}$$

then the two optimality and the envelope condition are given by

$$l_{\alpha,t}: \quad (Q^w_{\alpha,t})^{\rho-1}\varphi'(\ell_{\alpha,t}) = \beta R_{t+1} G^{\rho-1}\bar{\lambda}^w_{\alpha,t+1}(1-\tau^W_t)w_t\theta_t, \tag{5.1.15}$$

$$C^r_{\alpha,t}: \quad (Q^w_{\alpha,t})^{\rho-1} = \beta R_{t+1} G^{\rho-1}\bar{\lambda}^w_{\alpha,t+1} pc_t, \tag{5.1.16}$$

$$A^w_{\alpha,t}: \quad \lambda^w_{\alpha,t} = \beta G^{\rho-1}\bar{\lambda}^w_{\alpha,t+1} R_{t+1}. \tag{5.1.17}$$

First, observe that combining (5.1.15) and (5.1.17) implies exactly the same labor supply function as in the Blanchard model

$$\varphi'(\ell_{\alpha,t}) \cdot pc_t = (1-\tau^W_t)w_t\theta_t \quad \Rightarrow \quad \ell_{\alpha,t} = \ell_t, \tag{5.1.18}$$

The participation decision also works analogously. Combine (5.1.16) and (5.1.17) to get $\lambda^w_{\alpha,t} = (Q^w_{\alpha,t})^{\rho-1}/pc_t$. Insert this into the definition of $\bar{\lambda}^w_{\alpha,t+1}$ to get

$$\bar{\lambda}^w_{\alpha,t+1} = 1/pc_{t+1}\Upsilon_{\alpha,t+1}\left[\omega Q^w_{\alpha,t+1} + (1-\omega)C^r_{\alpha',t+1}\Lambda_{\alpha,t+1}\right]^{\rho-1}, \tag{5.1.19}$$

where $\Lambda_{\alpha,t+1} = \frac{V^r_{\alpha',t+1}/C^r_{\alpha',t+1}}{V^w_{\alpha,t+1}/Q^w_{\alpha,t+1}}$ and $\Upsilon_{\alpha,t+1} = \omega + (1-\omega)\left(\Lambda_{\alpha,t+1}\right)^{1-\rho}$. We can now eliminate the shadow values in the envelope condition to get the modified Euler equation for the worker that incorporates the chance of retirement

$$G\left[\omega Q^w_{\alpha,t+1} + (1-\omega)C^r_{\alpha',t+1}\Lambda_{\alpha,t+1}\right] = \left[\beta R_{t+1}\Upsilon_{\alpha,t+1}\frac{pc_t}{pc_{t+1}}\right]^{\sigma} Q^w_{\alpha,t}. \tag{5.1.20}$$

Observe that this expression nests the Euler equation of the Blanchard model for $\omega = 1$, i.e. if there was no retirement.

Proposition 5.1.1. *Solution first part.*

$$\begin{aligned}
(i) \quad & V^w_{\alpha,t} = (\Omega^w_t)^{1/\rho} \cdot Q^w_{\alpha,t}, \\
(ii) \quad & V^r_{\alpha,t} = (\Omega^r_t)^{1/\rho} \cdot C^r_{\alpha,t}, \\
(iii) \quad & \Omega^w_t = 1 + \gamma^w_{t+1}\beta \left(R_{t+1}\Upsilon^w_{t+1}\frac{pc_t}{pc_{t+1}} \right)^{\sigma-1} \Omega^w_{t+1}, \\
(iv) \quad & \Omega^r_t = 1 + \gamma^r_{t+1}\beta \left(R_{t+1}\frac{pc_t}{pc_{t+1}} \right)^{\sigma-1} \Omega^r_{t+1}.
\end{aligned}$$

Proof. The idea of the proof is to show that starting from the Euler equation (5.1.20) that (i) and (iii) will fulfill the Bellman equation (5.1.12). Using analogous steps have to be taken to prove (ii) and (iv) for the retirees. First, observe that the definition of the inverse marginal propensity to consume Ω in (iii) implies that it is independent of history α. Second, insert (i) in the definition of $\Lambda_{\alpha,t+1}$ to get $(\Omega^r_{t+1}/\Omega^w_{t+1})^{1/\rho}$ which is consequently also independent of α. Use this to eliminate Λ_{t+1} in the Euler equation (5.1.12), replace the consumption terms using (i) and (ii), multiply by Ω^w_{t+1} and use the definition of $\bar{V}^w_{\alpha,t+1}$ to get

$$G\bar{V}^w_{\alpha,t+1} = \left[\beta R_{t+1}\Upsilon_{\alpha,t+1}\frac{pc_t}{pc_{t+1}}\right]^{\sigma} V^w_{\alpha,t} \cdot (\Omega^w_{t+1}/\Omega^w_t)^{1/\rho}.$$

Raise the equation to the power of ρ, multiply with $\gamma^w_{t+1}\beta$ and use $\sigma\rho = \sigma - 1$. Replace the term $\gamma^w_{t+1}\beta^{\sigma}\left[R_{t+1}\Upsilon_{\alpha,t+1}\frac{pc_t}{pc_{t+1}}\right]^{\sigma-1}\Omega^w_{t+1}$ with $\Omega^w_t - 1$ using (ii) to arrive at

$$\gamma^w_{t+1}\beta G\bar{V}^w_{\alpha,t+1} = V^w_{\alpha,t} \cdot (\Omega^w_t - 1)/\Omega^w_t.$$

Use (i) again on the right-hand side to establish that the Bellman equation (5.1.12) is fulfilled. ∎

Proposition 5.1.2. *Solution second part.*

$$\begin{aligned}
(i) \quad & Q^w_{\alpha,t} = (\Omega^w_t pc_t)^{-1} \cdot \left[A^w_t + H^w_t\right], \\
(ii) \quad & C^r_{\alpha,t} = (\Omega^r_t pc_t)^{-1} \cdot \left[A^r_t + H^r_t\right], \\
(iii) \quad & H^w_{\alpha,t} = \bar{y}_t + \gamma^w_{t+1}G\bar{H}^w_{\alpha,t+1}/(\Upsilon_{t+1}R_{t+1}), \\
& \textit{where} \;\; \bar{H}^w_{\alpha,t+1} = \left[\omega H^w_{\alpha,t+1} + (1-\omega)H^r_{\alpha',t+1}(\Lambda_{t+1})^{1-\rho}\right], \\
(iv) \quad & H^r_{\alpha,t} = y^r_t + \gamma^r_{t+1}GH^r_{\alpha,t+1}/R_{t+1}.
\end{aligned}$$

Proof. The proof is done by showing that consumption function (i) is consistent with the Euler equation (5.1.20). We rely on the results of the first part to the solution above. Again the same steps have to be taken for (ii) and the retirees. Insert (i) and (ii) on the left-hand side of the Euler equation (5.1.20) to eliminate the consumption terms, collect terms, use the definition of Υ_{t+1} and $\bar{H}^w_{\alpha,t+1}$ and use $A^r_{\alpha',t+1} = A^w_{\alpha,t+1}$ to get

$$G\left(\Upsilon_{t+1}A^w_{\alpha,t+1} + \bar{H}^w_{\alpha,t+1}\right)/(\Omega^w_{t+1}pc_{t+1}) = \left[\beta R_{t+1}\Upsilon_{t+1}\frac{pc_t}{pc_{t+1}}\right]^{\sigma} Q^w_{\alpha,t}$$

Multiply by $\gamma^w_{t+1}\Omega^w_{t+1}pc_{t+1}/(R_{t+1}\Upsilon_{t+1})$ and replace $\gamma^w_{t+1}GA^w_{\alpha,t+1}/R_{t+1}$ by $A^w_{\alpha,t}+y^w_t - pc_tC^w_{\alpha,t}$ using (5.1.4). Next replace $\gamma^w_{t+1}G\bar{H}^w_{\alpha,t+1}/(\Upsilon_{t+1}R_{t+1})$ by $H^w_{\alpha,t} - \bar{y}_t$ by using (iii) and arrive at

$$A^w_{\alpha,t} + y^w_t - pc_tC^w_{\alpha,t} + H^w_{\alpha,t} - \bar{y}_t = \gamma^w_{t+1}\beta^{\sigma}\left[R_{t+1}\Upsilon_{t+1}\frac{pc_t}{pc_{t+1}}\right]^{\sigma-1}\Omega^w_{t+1}pc_tQ^w_{\alpha,t}.$$

Now use (5.1.13) to rewrite the left-hand side as $A^w_{\alpha,t} + H^w_{\alpha,t} - pc_tQ^w_{\alpha,t}$. Next, rearrange to get

$$(A^w_t + H^w_t)/(pc_tQ^w_t) = 1 + \gamma^w_{t+1}\beta^{\sigma}\left[R_{t+1}\Upsilon_{t+1}\frac{pc_t}{pc_{t+1}}\right]^{\sigma-1}\Omega^w_{t+1}.$$

Observe that the right-hand side is equal to Ω^w_t which leaves us with the consumption function (i). ∎

5.1.4 Aggregation

Aggregation works analogously to the sections before. Again, as labor income is independent of history α so will be $H^a_{\alpha,t}$, which we simply denote as h^a_t. Hence, aggregate human wealth is $H^a_t = h^a_tN^a_t$ for $a \in \{w, r\}$. The aggregate laws of motion are therefore given as follows

$$C_t^r = (\Omega_t^r pc_t)^{-1} (A_t^r + H_t^r), \tag{5.1.21}$$

$$Q_t^w = (\Omega_t^w pc_t)^{-1} (A_t^w + H_t^w), \tag{5.1.22}$$

$$GA_{t+1}^w = R_{t+1}\omega S_t, \quad S_t = [A_t^w + y_t^w N_t^w - pc_t C_t^w], \tag{5.1.23}$$

$$GA_{t+1}^r = R_{t+1}\left[A_t^2 + y_t^2 N_t^2 - pc_t C_t^2 + (1-\omega)S_t\right], \tag{5.1.24}$$

$$h_t^r = y_t^r + \gamma_{t+1}^r \frac{Gh_{t+1}^r}{R_{t+1}}, \tag{5.1.25}$$

$$h_t^w = \bar{y}_t + \gamma_{t+1}^w \frac{G\left[\omega h_{t+1}^w + (1-\omega)h_{t+1}^r(\Lambda_{t+1})^{1-\rho}\right]}{\Upsilon_{t+1}R_{t+1}} \tag{5.1.26}$$

5.1.5 Production

Production is identical to the case of the Blanchard model with government, see section 4.6.

5.1.6 Government

Government is very similar to the Blanchard model with government, see section 4.6.2. Government expenditure is given as

$$Exp_t = C_t^G + (1-\delta_t^w)b_t N_t^w + P_t N_t^r, \tag{5.1.27}$$

where the second term is total welfare benefits paid and the third term is the expenditure on old age pensions. Revenues are

$$Rev_t = T_t^F + \left(\tau_t^F L_t^D + \tau_t^W L_t^S\right) w_t + \tau_t^l N_t + \tau_t^C C_t. \tag{5.1.28}$$

The primary balance is again the difference of revenues and expenditures.

5.1.7 Exercises

Exercises

Ex. 10 — The Gertler Model

Implement the two-age-group Gertler model based on the code

`Blanchard_government`. Pick reasonable values for the additional parameters.

Ex. 11 — Life-Cycle Savings Decision in Presence of a Pension System
Based on the implementation of the Gertler model (Ex. 10), simulate a complete cut of pension benefits. Compare savings behavior (marginal propensities to consume) of young versus old over time and interpret.

5.2 The Probabilistic Aging Model

The probabilistic aging model (Grafenhofer et al. (2007)) is a natural extension of the Gertler model from 2 to A age groups which is briefly discussed now. Let a be again the age group index with $a \in \{1, 2, ..., A\}$. This implies the following laws of motion for population

$$\begin{aligned}
&\text{death:} && N^{\dagger}_{\alpha,t+1} = N^{a}_{\alpha,t} \cdot (1 - \gamma^{a}_{t+1}), \\
&\text{no aging:} && N^{a}_{\alpha,t+1} = N^{a}_{\alpha,t} \cdot \gamma^{a}_{t+1}\omega^{a}, \\
&\text{aging:} && N^{a+1}_{\alpha',t+1} = N^{a}_{\alpha,t} \cdot \gamma^{a}_{t+1}(1 - \omega^{a}).
\end{aligned}$$

where $1 - \omega^a$ is set to reflect the average time spent in age group a before aging into age group $a+1$. An advantage of this specification is that the size of the age groups can be detached from the frequency of the model, i.e. the model can be used at yearly frequency without the need of using single-year age groups as in the Auerbach-Kotlikoff model (chapter 6), while in contrast to the Gertler model a more realistic life-cycle behavior can be modeled. The Probabilistic Aging model is typically implemented using around 10 age groups, where a cut-off age group a^R is defined such that households with $a < a^R$ are workers whose problem is similar to the one of the workers in the Gertler model and households with $a > a^R$ face a similar decision problem as the retirees in the Gertler model. A novelty is the use of a mixed age group $a = a^R$, where the participation decision is interpreted as the retirement decision as illustrated by the expected per-period incomes

$$y^{a}_{\alpha,t} = \begin{cases} \delta^{a}_{\alpha,t}(1 - \tau^{W,a}_{t})w^{a}_{t}\ell^{a}_{\alpha,t}\theta^{a}_{t} + (1 - \delta^{a}_{\alpha,t})b^{a}_{t} - \tau^{l}_{t} & \text{if } a < a^R \\ \delta^{a}_{\alpha,t}(1 - \tau^{W,a}_{t})w^{a}_{t}\ell^{a}_{\alpha,t}\theta^{a}_{t} + (1 - \delta^{a}_{\alpha,t})P^{a}_{t} - \tau^{l}_{t} & \text{if } a = a^R \\ P^{a}_{t} - \tau^{l}_{t} & \text{if } a > a^R. \end{cases} \tag{5.2.1}$$

Chapter 6

OLG - The Auerbach-Kotlikoff Model

Summary

This section introduces the reader to the Auerbach-Kotlikoff model with mortality. In contrast to the Blanchard model we now explicitly keep track of every age class which also allows for a much higher level of realism as every age class can be parameterized individually but also is computationally more challenging. This allows to match life-cycle profiles of income and consumption and makes the model a powerful tool to address questions of inter-generational redistribution. On top of the public policy instruments introduced in the previous section we model a simplified pay-as-you-go (PAYG) pension system. The presence of mortality shocks demands explicit rules of how left-over assets are distributed over the remaining population. The corresponding code is `AuerbachKotlikoff`.

6.1 Small Open Economy

In summary the following government instruments are considered in this section. Note that we explicitly differentiate between pension contributions and other wage dependent taxes (or other contributions)

- Pension contributions from worker: $\tau^{W,c}$
- Income taxes from workers and retirees: $\tau^{W,i}$

- Pension contributions from firm: $\tau^{F,c}$
- Other pay-roll taxes from firms: $\tau^{F,i}$
- Total tax burden on labor for worker[1]: $\tau^{W} = \tau^{W,i} + \tau^{W,c} - \nu \cdot \tau^{W,i} \cdot \tau^{W,c}$
- Total tax burden on pay-roll for firms: $\tau^{F} = \tau^{F,i} + \tau^{F,c}$
- Lump-sum taxes/transfers from/to households: τ^{l}
- Profit taxes from firms: τ^{prof}
- Profit tax deductibility options for capital maintenance costs: ϕ_0^{τ}
- Consumption taxes: τ^{C}
- Unproductive government consumption: C^{G}
- Benefits for non-participating households: b
- Gross pension payments: P
- The effective retirement age: reflected in ϕ

6.1.1 Demography

The model works with $A+1$ representative households that differ by age a, i.e. $a \in \{0, ..., A\}$.[2] Aging is directly linked to the evolution of time indexed with t, i.e. a household of age a at time t is of age $a+n$ in $t+n$ (unless it died in the meantime). Households die end of period t with an age-specific mortality rate $(1-\gamma_{t+1}^{a})$. Let N_t^a be the mass of households of age a at time t and NB_{t+1} the number of newborns end of period t. The demographic structure is defined by a simple system of equations

$$N_{t+1}^{0} = NB_{t+1} \tag{6.1.1}$$

$$N_{t+1}^{a+1} = \gamma_{t+1}^{a} N_t^{a}, \quad \gamma_t^{A} = 0 \ \forall t. \tag{6.1.2}$$

[1]The parameter ν nests different specification depending on whether pension contributions are income tax deductible or not.

[2]Note that $a=0$ does not necessarily have to be the beginning of life, it could as well be the beginning of adulthood.

The restriction $\gamma^A = 0$ guarantees that the maximum attainable age is A. Total population size is given by

$$N_t = \sum_{a=0}^{A} N_t^a. \tag{6.1.3}$$

6.1.2 Households

Households face the following intertemporal budget constraint (6.1.4). Observe that we did not assume a reverse-life insurance as in the Blanchard model before. More realistically, we assume that asset holdings of households that died because of a mortality shock are bequeathed to younger age groups. Note that the end of period asset holdings of households of maximum age A are zero because they die with certainty. ab_t^a denote the flow incomes of younger households from these **accidental bequests**. The fact that the individual law of motion for assets is not directly affected will have important implications on the consumption profile over age.

$$GA_{t+1}^{a+1} = R_{t+1}\left[A_t^a + \bar{y}_t^a - pc_t C_t^a\right], \quad \bar{y}_t^a = y_t^a + iv_t^a + ab_t^a \tag{6.1.4}$$

where $\bar{y}_t^a$ denotes total per period income flows. iv^a are intervivo transfers between different generations with the condition that $\sum_{a=0}^{A} iv_t^a N_t^a = 0$, i.e. iv_t^a can be positive for some and negative for other age groups. This is introduced in order to fit the model better to age-specific asset profiles in the implementation. Per period income (without intervivo transfers and accidental bequests) y_t^a is given by

$$y_t^a = \phi_t^a\left[\delta_t^a(1-\tau_t^{W,a})w_t\ell_t^a\theta_t^a + (1-\delta_t^a)b_t^a\right] + (1-\phi_t^a)\,(1-\tau^{W,i,a})P_t^a - \tau_t^{l,a}. \tag{6.1.5}$$

This requires some explanation. First, participation works like in the previous section with the exception that it is age-specific, i.e.

$$\underline{h}_t^a = \left[(1-\tau_t^{W,a})w_t\ell_t^a\theta_t^a - b_t^a\right]/pc_t - \varphi^a(\ell_t^a), \;\; \Rightarrow \;\; \delta_t^a = F^a(\underline{h}_t^a). \tag{6.1.6}$$

The participation decision and the hours decision however only occur when a household is not retired. ϕ_t^a denotes an indicator variable whether a household is retired or not. Hence, the first term is labor income which is effective labor supply, i.e. hours ℓ_t^a times age-specific productivity θ_t^a times after tax and contribution wage rate $(1-\tau_t^{W,a})w_t$ for the time spent on the labor market. Note that the wage rate w_t per effective unit of labor supply is not age dependent as workers of all age groups interact in a single labor market. During non-participation households get benefits b^a and do not pay taxes nor contributions (by assumption). The second term of (6.1.5) denotes the net income in case of retirement where $(1-\tau^{W,i,a})P^a$ is an after tax pay-as-you-go pension, i.e. financed out of current pension contributions from workers and the firm. Note that retirees themselves do not pay contributions to the system anymore. While in principle ϕ_t^a can only take the values 0 and 1 we allow that it can take intermediate values for one age group in order to reflect retirement during a calendar year. For simplicity we treat the retirement as an exogenous policy parameter assuming that the government can directly control effective retirement.[3] The last term $\tau_t^{l,a}$ reflects lump-sum taxes (or transfers if negative) which do not alter the labor supply decision. The disutility of providing labor supply and the value of home production are included in the consumption bundle.[4] Hence, the conditional expectation of the value of home production during non-retirement is again $h^{a,e} = F^a(\underline{h}^a)^{-1}\int_{-\infty}^{\underline{h}^a} h\, dF^a(h)$. By assumption retired workers receive no value from home production. The optimization problem of an individual household of age a looks as follows

$$V(A_t^a) = \max_{C_t^a,\ell_t^a} \quad \left[{}^1\!/\!_\rho\,(Q_t^a)^\rho + \beta\gamma_{t+1}^a G^\rho V_{t+1}^{a+1}\right], \qquad \text{s.t.} \tag{6.1.7}$$

$$(6.1.4),\ (6.1.5)\ \text{and} \tag{6.1.8}$$

$$Q_t^a = C_t^a - \Psi_t^a, \quad \Psi_t^a = \phi_t^a\left[\delta_t^a\varphi^a\left(\ell_t^a\right) - (1-\delta_t^a)h_t^{a,e}\right].$$

Define the change in remaining life utility at time t to a marginal increase in financial wealth as $\lambda_t^a \equiv \partial V_t^a/\partial A_t^a$. The two optimality and the envelope

[3]Endogenous retirement in Auerbach-Kotlikoff models exists but comes with numerical challenges as it involves discrete optimization.

[4]The disutility function itself can also be age dependent, e.g. to capture different labor supply elasticities.

conditions are

$$\begin{aligned} C_t^a: &\quad (Q_t^a)^{\rho-1} = \gamma_{t+1}^a \beta G^{\rho-1} R_{t+1} \lambda_{t+1}^{a+1} pc_t && (6.1.9) \\ \ell_t^a: &\quad (Q_t^a)^{\rho-1}\, \phi_t^a \delta_t^a \varphi^{a\prime}(\ell_t^a) = \gamma_{t+1}^a \beta G^{\rho-1} R_{t+1} \lambda_{t+1}^{a+1} \phi_t^a \delta_t^a (1-\tau_t^{W,a}) w_t \theta_t^a && (6.1.10) \\ A_t^a: &\quad \lambda_t^a = \gamma_{t+1}^a \beta G^{\rho-1} R_{t+1} \lambda_{t+1}^{a+1} && (6.1.11) \end{aligned}$$

Combine (6.1.9) and (6.1.11) to get $(Q_t^a)^{\rho-1} = \lambda_t^a pc_t$. Use this expression again in (6.1.11) to derive the Euler equation

$$GQ_{t+1}^{a+1} = \left[\frac{pc_t}{pc_{t+1}} \gamma_{t+1}^a \beta R_{t+1}\right]^\sigma Q_t^a. \tag{6.1.12}$$

This looks very similar to the Euler equations we have studied in the previous sections with one important exception. The survival rate γ_{t+1}^a enters this expression. This is only due to dropping the reverse-life insurance assumption and is no peculiarity of the Auerbach-Kotlikoff model as such. Let us for the moment assume that consumption prices are constant over time. Hence, the choice of the consumption bundle Q^a will increase in age if $[\gamma_{t+1}^a \beta R_{t+1}]^\sigma / G > 1$. As the survival rate γ_{t+1}^a decreases in age there can be situations where $[\gamma_{t+1}^a \beta R_{t+1}]^\sigma / G > 1$ for young age groups but $[\gamma_{t+1}^a \beta R_{t+1}]^\sigma / G < 1$ for older households. This would give rise to a hump-shaped consumption profile (as observed in the data). We will revisit this topic again in the implementation section 6.2. Next, combine (6.1.10) and (6.1.9) to get a simple implicit expression of optimal labor supply

$$pc_t \varphi^{a\prime}(\ell_t^a) = (1-\tau_t^{W,a}) w_t \theta_t^a. \tag{6.1.13}$$

Using equivalent steps as in the sections before one can derive the characterization of the consumption function for a household of age a

$$Q_t^a = (\Omega_t^a pc_t)^{-1} [A_t^a + H_t^a], \quad (6.1.14)$$

$$\Omega_t^a = 1 + \left(\gamma_{t+1}^a \beta\right)^\sigma \left(R_{t+1} \frac{pc_t}{pc_{t+1}}\right)^{\sigma-1} \Omega_{t+1}^{a+1}, \quad (6.1.15)$$

$$H_t^a = \bar{y}_t^a - \Psi_t^a pc_t + \frac{GH_{t+1}^{a+1}}{R_{t+1}}, \quad (6.1.16)$$

$$C_t^a = Q_t^a + \Psi_t^a. \quad (6.1.17)$$

Observe the important difference to the Blanchard model discussed before. Because some of the variables, e.g. the survival rates or the income flows, differ by age we can no longer analytically aggregate and describe the economy as governed by a single consumption function. Instead we have to solve the household problem for every age group individually and can only aggregate the results afterwards.

6.1.3 Accidental Bequests

We now have to specify the conditions received accidental bequests ab_t^a have to fulfill. We assume the following timing. At the beginning of a period households receive accidental bequests at the same time as the other income flows and start consuming. At the end of the period some households die and leave their savings S_t^a,

$$S_t^a = A_t^a + y_t^a + iv_t^a + ab_t^a - pc_t C_t^a. \quad (6.1.18)$$

This implies that at the end of every the period the following total assets are collected $\sum_{a=0}^{A}(1-\gamma_{t+1}^a) S_t^a N_t^a$. Hence, the following condition which equates accidental assets received and given has to hold

$$\sum_{a=0}^{A} ab_t^a N_t^a = \sum_{a=0}^{A}(1-\gamma_{t+1}^a) S_t^a N_t^a. \quad (6.1.19)$$

More specifically, one can assume a distribution rule of the following form

$$ab_t^a = \xi_t^a \cdot \frac{\sum_{a=0}^{A}(1-\gamma_{t+1}^a)S_t^a N_t^a}{N_t^a}, \qquad \sum_{a=0}^{A} \xi_t^a = 1, \tag{6.1.20}$$

where ξ_t^a denote some exogenous weights.

6.1.4 Aggregation

Aggregate effective labor supply is $L_t^S = \sum_{a=0}^{A} \phi_t^a \delta_t^a \ell_t^a \theta_t^a N_t^a$. It is handy to define the average labor and contribution tax rates over all ages paid by the workers as $\tau_t^W = \frac{\sum_{a=0}^{A} \tau_t^{W,a} \phi_t^a \delta_t^a \ell_t^a \theta_t^a N_t^a}{L_t^S}$. This way one can conveniently write aggregate after tax and contribution labor income as $w_t(1-\tau_t^W)L_t^S$ like we did in the Blanchard model. Similarly, $\tau^{W,c}$ is used as notation for the average pension contribution rate. All other variables are aggregated by adding all age classes, i.e. for some variable X we define

$$X_t = \sum_{a=0}^{A} X_t^a N_t^a. \tag{6.1.21}$$

The evolution of aggregated assets can be computed as follows.[5] Multiply (6.1.4) with $N_t^a \gamma_{t+1}^a$, use the demographic law of motion (6.1.2), sum over a

[5] The law of motion of aggregate assets is of much less value for us than it was in the Blanchard model, because we explicitly have to solve for all age-specific asset holdings anyway.

and eliminate $\sum_{a=0}^{A} ab_t^a N_t^a$ using (6.1.19).

$$\begin{aligned}
GA_{t+1}^{a+1}N_t^a =& R_{t+1}S_t^a N_t^a \Leftrightarrow GA_{t+1}^{a+1}N_{t+1}^{a+1} = R_{t+1}\gamma_{t+1}^a S_t^a N_t^a \Leftrightarrow \\
GA_{t+1}^{a+1}N_{t+1}^{a+1} =& R_{t+1}\left(S_t^a - (1-\gamma_{t+1}^a)S_t^a\right)N_t^a \Rightarrow \\
G\sum_{a=0}^{A} A_{t+1}^{a+1}N_{t+1}^{a+1} =& R_{t+1}\sum_{a=0}^{A}\left(S_t^a - (1-\gamma_{t+1}^a)S_t^a\right)N_t^a \Leftrightarrow \\
G\sum_{a=0}^{A} A_{t+1}^{a+1}N_{t+1}^{a+1} =& R_{t+1}\sum_{a=0}^{A} S_t^a N_t^a - R_{t+1}\sum_{a=0}^{A} ab_t^a N_t^a \Leftrightarrow \\
G\sum_{a=0}^{A} A_{t+1}^{a+1}N_{t+1}^{a+1} =& R_{t+1}\sum_{a=0}^{A}\left[A_t^a + y_t^a + iv_t^a - pc_t C_t^a\right]N_t^a \Leftrightarrow \\
G\sum_{a=0}^{A} A_{t+1}^{a+1}N_{t+1}^{a+1} =& R_{t+1}\left[A_t + y_t + iv_t - pc_t C_t\right]
\end{aligned}$$

The left hand side can be rearrange as follows using the fact that 'newborns' have zero assets as do people in the last age group at the end of the period.

$$\begin{aligned}
G\sum_{a=0}^{A} A_{t+1}^{a+1}N_{t+1}^{a+1} =& G\sum_{a=-1}^{A-1} A_{t+1}^{a+1}N_{t+1}^{a+1} + G\underbrace{A_{t+1}^{A+1}N_t^{A+1}}_{=0} - G\underbrace{A_{t+1}^{0}N_t^{0}}_{=0} \\
=& G\sum_{a=-1}^{A-1} A_{t+1}^{a+1}N_{t+1}^{a+1} = G\sum_{a=0}^{A} A_{t+1}^{a}N_{t+1}^{a} = GA_{t+1}
\end{aligned}$$

Hence,

$$GA_{t+1} = R_{t+1}\left[A_t + y_t + iv_t - pc_t C_t\right]. \tag{6.1.22}$$

6.1.5 Production

Production is identical to the Blanchard model with government, see section 4.6.2.

6.1.6 Government

Government is very similar to the Blanchard model with government, see section 4.6.2. Government expenditure is given as

$$Exp_t = C_t^G + \sum_{a=0}^{A} \phi_t^a(1-\delta_t^a)b_t^a N_t^a + \sum_{a=0}^{A}(1-\phi_t^a)P_t^a N_t^a, \tag{6.1.23}$$

where the second term is total welfare benefits paid and the third term is the expenditure on old age pensions. Revenues are

$$Rev_t = T_t^F + \left(\tau_t^F L_t^D + \tau_t^W L_t^S\right) w_t + \tau_t^l N_t + \tau_t^C C_t + \sum_{a=0}^{A}(1-\phi_t^a)\tau_t^{W,i,a} P_t^a N_t^a. \tag{6.1.24}$$

The primary balance is again the difference of revenues and expenditures. The overall budget can be divided into the pension system and the general budget. Payments into the pension system are the sum of all contributions, i.e. $\left(\tau_t^{F,c} L_t^D + \tau_t^{F,c} L_t^S\right) w_t$ while the expenditure is given by $\sum_{a=0}^{A}(1-\phi_t^a)P_t^a N_t^a$. If contributions fall short of the pension payouts the system is implicitly cross subsidized by the general budget.

6.1.7 Equilibrium and Walras' Law

Computation of both temporary equilibrium and the steady state follows the same procedure as thoroughly explained in section 4.6. The only difference is that the household decisions have to be computed $A+1$ times instead of once and then have to be aggregated to fulfill the market clearing conditions (4.6.44) to (4.6.47). In every period Ω^a and H^a have to be solved backward in age, while afterwards A^a and Q^a have to be solved forward in age. For the derivation of Walras' Law we define the excess demand for intervivo-transfers as $\zeta_t^{IV} = -iv_t$ and the excess demand for accidental bequest as $\zeta_t^{AB} = \sum_{a=0}^{A}(1-\gamma_{t+1}^a)S_t^a N_t^a - ab_t$. Insert both in (6.1.22) and follow the steps of the proof in section (4.6) to establish

$$\zeta_t^Y + w_t\zeta_t^L + \zeta_t^A + \zeta_t^G + \zeta_t^{IV} + \zeta_t^{AB} - \frac{G\zeta_{t+1}^A}{R_{t+1}} = 0. \tag{6.1.25}$$

Consequently, the steady state version of Walras' Law is

$$\zeta^Y + w\zeta^L + \zeta^G + \zeta^{IV} + \zeta_t^{AB} + \frac{r-g}{R}\zeta^A = 0. \tag{6.1.26}$$

6.1.8 Intergenerational Distribution

The Auerbach-Kotlikoff model is an obvious tool to evaluate redistributional effects of policy reforms between generations. Keeping track of individual consumption bundles for different age groups and different points in time allows us to compute the indirect utility of remaining life time for different ages and dates. Indirect utility can be computed by recursively inserting the optimal solutions for consumption bundle Q into (6.1.7). One can then evaluate a reform by computing welfare changes in (pure) consumption equivalent terms. For that we compute by which constant factor every C has to be multiplied for the remaining life time to end up with the same indirect utility. This change can be compared to the initial steady state indirect utility values where, by construction, all generations have the same indirect utility for a given age. One should however always keep in mind that welfare gains from insurance against myopic behavior (one of the justifications of having a public pension system in the first place) are not included in our welfare measures. In the codes for the Auerbach-Kotlikoff model (with and without earnings related pensions, for the latter see section 6.4.1) `AuerbachKotlikoff` and `AuerbachKotlikoff_earnings_link` measurement of welfare changes is done in the functions `welfare` (relative change in indirect utility) and `welfareC` (relative change in consumption equivalent terms).

6.2 Calibration and Implementation

Calibrating models like those described so far typically comes with two type of variables that have to be treated differently in the calibration. One type is set in order to replicate first moments of the data in the initial steady state. The other type is chosen in order to capture the correct 'behavior' of the model (e.g. elasticities, etc.). The latter type has be discussed e.g. in the section on micro- vs. macro-elasticity of labor supply. For the moment matching parameters we can distinguish those that are simply set to their empirical values, e.g. observed interest rate or the labor productivity growth rate and those that have no clear (or a non or hardly observable) empirical counterpart which are chosen in order to replicate another empirical target. An example would be setting the scaling factor in the production function in order to match (normalized) GDP. As we typically want to look at structural reforms we want to eliminate any influence of the current position in the business cycle on our calibration. Hence, for variables that are particularly sensitive to the business cycle a simple way is to take an average over the last couple of years instead of just using the very last observation (e.g. for the real interest rate). The task of finding an appropriate calibration target for a parameter is far from trivial and depends on the type of analysis that should be carried out. An example is the discount rate ρ. On one hand it can be used to replicate the approximate shape of life-cycle consumption. On the other hand it might be used to replicate the average trade balance. Depending on which target is chosen this might imply very different values for ρ. In principle there is often a choice to be made on whether a target at the household level or the economy level should be matched. Discrepancies between those two can occur due to simplistic modeling or if the data is taken from a non-stationary economy. An example for the former is matching the tax burden at the household level where the modeler would want to capture the effective tax at the margin versus matching total tax revenue. If the tax system is approximated in a linear way outcomes are very likely to be different depending on the calibration target. An example for the calibration target trade-off due to non-stationary of the data is replicating different aspects of the demography. This will be explained in more detail in the next section. For an efficient implementation it is convenient to find the indirect calibration targets (data moments) numerically using a root-finding algorithm (do not set the corresponding parameter values by hand using trial-and-error!). Only then as the last step if the stationary economy is replicated in the calibration we adjust the behavior of the model to shocks (e.g. matching macro-elasticities or the recovery time of the capital stock after a shock, etc.). Table 6.2.1 holds an incomplete matching of parameters and calibration targets before we discuss some critical calibration challenges in more detail.

6.2.1 Calibration of Demography

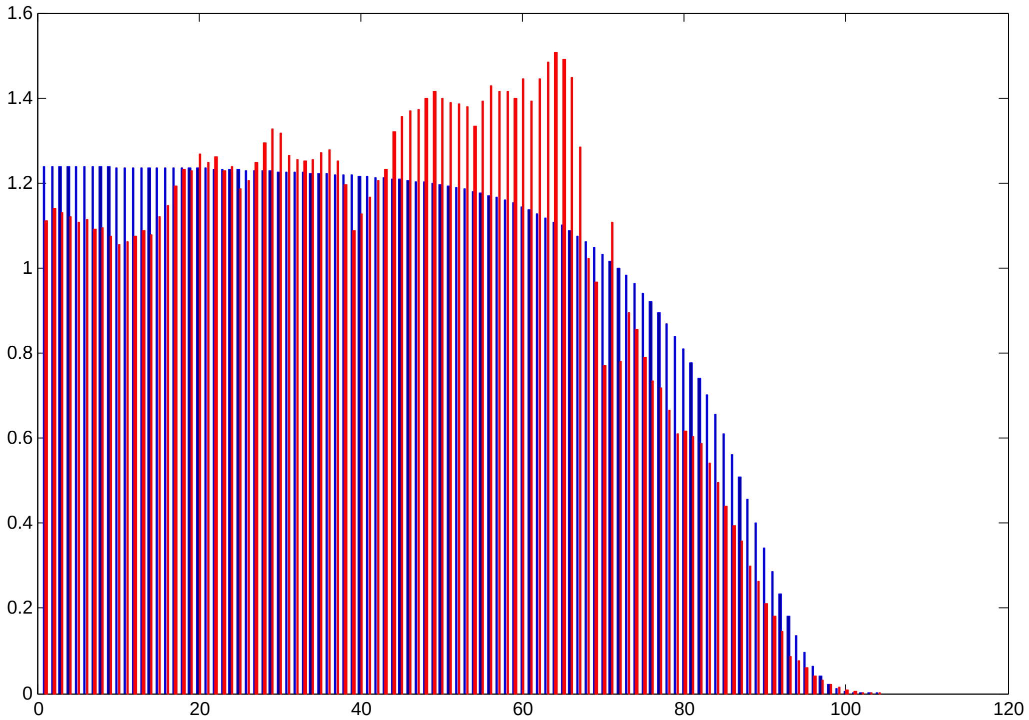
Calibrating the demography typically comes with the challenge that the currently observed demographic distribution is in transition and not stationary. Given the current mortality rates the population is younger than in a final steady state (assuming that mortality rates are frozen). We illustrate this for Finland. Eurostat publishes the population mass for every age as well as life tables that contain the conditional mortality rates, i.e. the empirical equivalent of $1-\gamma^a$. Using data for 2010 the dependency ratio is 0.244 using the 2010 population shares while it would be 0.290 for a stationary population.

Table 6.2.1: Matching calibration targets

Variable/Parameter	Symbol	Code Name	Target
GDP	Y	`Y0`	normalization
population size	N	`N0`	normalization
number of age groups	A	`nag`	modeling choice
effective retirement age	-	`nwg0`	data
discount rate	$(1-\beta)/\beta$	`rho`	data, match consumption profile/ trade balance
labor productivity growth	g	`g`	data
depreciation rate	δ^K	`delta`	data
intertemporal elasticity of substitution	σ	`sigma`	literature
elasticity hours	ε_ℓ	`epsl`	literature, micro-elasticity
elasticity participation	κ	`epsp`	literature, micro-elasticity
labor share	$1-\alpha$	`1-alpha`	data
productivity profiles	θ^a	`thetav0`	data, match income profile
scaling factor capital adjustment costs	ψ	`psi`	literature, match recovery speed
hours supply	ℓ^a	`ellv0`	data, matched by setting φ_0^a
participation rates	δ^a	`partv0`	data, matched by setting k^a
benefits	b^a	`bv0`	data, match aggregate expenditure
pension payments	P^a	`pv0`	data, match aggregate expenditure
government debt	D^G	`DG0`	data
tax rates	$\tau^{W,i,a}, \tau^{W,c,a}, \ldots$	`tauWiv0`, `tauWcv0`,...	data, match aggregate revenues
government consumption	cG	`cG0`	budget closure
lump-sum taxes/transfers	τ^l	`taul0`	data, match aggregate private consumption
mortality	γ^a	`gamv0`	data
newborns	NB	`NB0`	data, population size
wage rate	w	`w0`	normalization
intervivo-transfers	iv^a	`ivv0`	data, match asset profile

We defined the dependency ratio as the number of 65+ aged persons divided by the number of persons younger than 65.[6] Using the empirical mortality rates therefore implies that we will overestimate the share of old persons in the population resulting in incorrect aggregate pension payments and tax revenues. On the other hand if we increased the mortality rates in order to match the actual dependency ratio this will lead to different consumption behavior, as households in the model would have a lower life expectancy than in reality. To illustrate the mismatch compare the actual age distribution (red bars) versus the stationary one implied by the actual mortality rates (blue bars) in figure 6.2.1. Next to the very old also the very young are

Figure 6.2.1: Fit of demography in calibration for Finland



Note: Population size and mortality data from Eurostat, 2010. Total population size normalized to 100. Blue bars reflect the predicted stationary distribution according to mortality rates from 2010. Red bars reflect the actual 2010 population size for different ages.

overestimated in the calibration. In contrast the share of persons between approximately 45 and 65 ('baby boomers') is gravely underestimated. In principle one can think of three options in order to improve the calibration.

[6]As we purely focus on the demography in this section below 20 year old persons are not excluded.

1. **Compromise between both targets**. This means that one tries to increase the observed mortality rates for the old persons prudently in order to come closer to the dependency ratio target without lowering life-expectancy too much. This can help to improve the fitting but can never replicate a case where $N^a < N^{a+1}$.

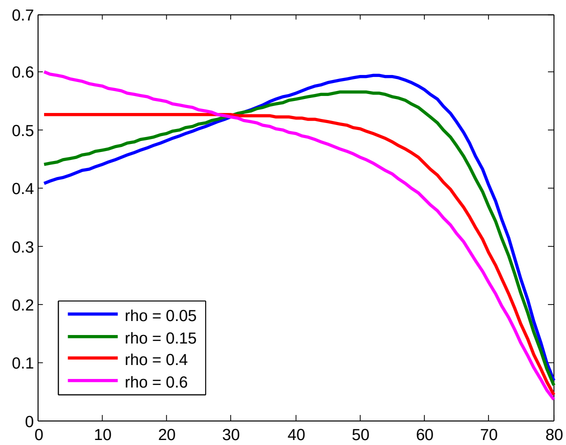
2. **Use 'pseudo'-migration flows**. When working with a simple implementation of migration to the model (see section 6.4.4) one can do the following. Calibrate the steady state migration flows such that the current population structure is perfectly matched. Once the simulation starts the actual migration flows are loaded (as a reform). The trick is simply to assume that the current observed population structure is not the outcome of a transitory process that is not completed yet but by a constant net migration pattern (unrelated to the observed migration flows), e.g. we assume that young persons constantly immigrate while old ones emigrate.

3. **Start demographic change before.** The cleanest solution is to start the demographic change already decades ago such that the current population structure is transitory in the model just like in reality. In order to evaluate a reform introduced now but not already anticipated in the early years when the simulation was started one can use the methods described in section 4.4. A drawback of this approach is how to precisely calibrate other variables, e.g. GDP, the private consumption and various tax shares for the current year if they are themselves just a transitory outcome.

6.2.2 Consumption and Asset Profiles

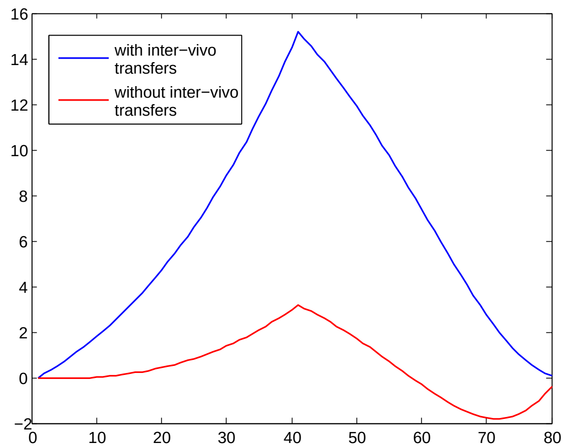
Empirical consumption profiles are usually hump-shaped in age.
A way to introduce this to a model with perfect foresight and consumption smoothing is with age-dependent mortality rates which will enter the Euler equation (as long as there is no reverse life insurance). For the given mortality rates and the observed real interest rate a hump-shaped consumption profile is only given for some choices of $\beta = 1/(1+\rho)$. Figure 6.2.2 illustrates this for different choices of the discount rate. The intuition is that in early years survival rates are very close to 1. We need ρ to be smaller than r in order to

Figure 6.2.2: Life-cycle consumption profile for different discount rates



have an increasing profile over the first years. Only when the mortality rates start getting larger consumption growth in the Euler equation falls below 1. The reasons for hump-shaped consumption profiles can be manifold. In a model where households are partially finance constrained (see section 6.4.3) the hump-shape can result from a similar shape to the income profile. An explicit warm-glow or bequest motive can also change the Euler equation in order to produce non-monotonic consumption-age profiles. The next thing to look at are the asset-age profiles.

Figure 6.2.3: Life-cycle asset profile with and without using intervivo transfers



With consumption smoothing and hump-shaped income profiles one would assume that households are net lenders, i.e. they have negative assets, during the first part of their lives. However, empirical profiles typically draw a picture of increasing (average) assets throughout life before only in the later years assets decline. One explanation could again be that many households face borrowing limits, hence, they cannot run into considerable debt while young. Another, is that young households receive asset transfers from older households. In the presented model this kind of intervivo transfers was introduced in an exogenous ad-hoc way for the single purpose of improving the fit of the asset-age profiles. Figure 6.2.2 illustrates the calibrated asset profile once with and once without using intervivo transfers.

6.2.3 Accounting

The available code provides a simple way of having an overview of the calibration and checking all identities. Use the function `writecalib2latex` after calibration to automatically produce the following output.[7]

Table 6.2.2: Production

Production cost			Absorption
Wage related income	72.000	Private consumption	50.000
Capital related income	28.000	Government consumption	37.187
		Capital investment	22.292
		Trade balance	-9.480
GDP	100.000	GDP	100.000

Table 6.2.3: Business Sector

Expenditure			Income
Wages	60.000	Output	100.000
Dividends	4.846		
Capital investment	22.292		
Corporate taxes	0.862		
Firm PAYG contributions	9.000		
Firm pay-roll taxes	3.000		
Total	100.000	Total	100.000

Table 6.2.4: Household Sector

Expenditure			Income
Private Consumption	50.000	Wages	60.000
Households PAYG contr.	6.000	Publ. transfers to households	-40.115
Income tax workers	2.700	PAYG Pensions	23.626
Income tax retirees	1.181	Non-part. benefits	10.602
Consumption taxes	10.000	Dividends	4.846
Accid. bequests given	13.065	Accid. bequests rec.	13.065
Intervivo trans. given	13.834	Intervivo trans. rec.	13.834
Private savings	6.553	Private interest earnings	17.475
Total	103.334	Total	103.334

Table 6.2.5: Government Sector

Expenditure			Income
Government Consumption	37.187	Income tax workers	2.700
Publ. transfers to households	-40.115	Income tax retirees	1.181
Non-part. benefits	10.602	Firm pay-roll taxes	3.000
Subsidy PAYG system	8.626	Corporate taxes	0.862
Interest payments	2.308	Consumption taxes	10.000
		Net fiscal deficit	0.865
Total	18.608	Total	18.608

Table 6.2.6: PAYG Pension System

Expenditure			Income
PAYG Pensions	23.626	Households PAYG contr.	6.000
		Firm PAYG contributions	9.000
		Subsidy PAYG system	8.626
Total	23.626	Total	23.626

Table 6.2.7: Current Account

Expenditure			Income
Net foreign savings	5.688	Trade balance	-9.480
		Foreign interest earnings	15.168
Total	5.688	Total	5.688

[7] Beware: The shown output was not calibrated to a specific country.

6.3 Exercises

Exercises

The goal is to produce a short policy advice (background) paper on the topic 'long-term public financing and aging'. Do this to the best of your knowledge given the limited time. The tasks are formulated deliberately in a vague way. Be creative. Put the focus on what you find most interesting.

Ex. 12 — Calibration
Calibrate the presented Auerbach-Kotlikoff-model (or the extension with earnings-related pensions) to an EU country of your choice. Keep data availability in mind before making your choice. Disclaimer: you are not required to work with micro-data. Nevertheless try to approximate age profiles for different variables as well as possible. Explain and document. If you cannot match certain targets explain how this might affect your results.

Ex. 13 — Aging simulation
Shock your model with an available population forecast that incorporates the changes in the age structure (e.g. from Eurostat, the UN or the respective national statistical offices). Hint: Because of block recursion use only the demography module first in order to replicate the predicted changes in the population structure as well as possible before doing the simulation of the 'economic part'. What are the consequences of aging in the medium and long run especially on public financing? Remark: Define a year close the to end of the projection horizon of the population forecast (e.g. 2060) as the long-run, even if the final steady state is set to a much later year e.g. 2300.

Ex. 14 — Reform simulation
Do three types of reforms (on top of Ex. 13) in order to cope with the financing difficulties of an aging society: (a) cut pension benefits, (b) raise pension contributions, and (c) raise the retirement age. Compare the outcome of the three measures in an appropriate way and explain your results.

Hint: For the interpretation of your results do not forget that the model is detrended.

6.4 Further Potential Extensions

In this section we introduce potential extensions and give a swift idea of how to model them. Thorough derivations are left as an exercise.

6.4.1 Earnings-Related Pension System

A possible extension concerning the pension system is to model the pension payments as a function of past earnings in contrast to a flat payout that everyone receives. As we will show an earnings-link has important implications already during the working life. In order to keep track of the earned pension rights we have to introduce an additional stock variable, P_t^a, which denotes the yearly gross pension payout (as in chapter 6.1). The law of motion for P^a is given as

$$GP_{t+1}^{a+1} = G_t^{P,a} \left[P_t^a + m_t^a \phi_t^a \delta_t^a w_t \ell_t^a \theta_t^a \right]. \tag{6.4.1}$$

m_t^a denotes the accumulation factor. It is age-dependent which allows to model different systems, e.g. where only the last couple of years' labor incomes matter in contrast to systems where the whole earnings history is taken into account. The factor $G_t^{P,a}$ determines the indexing of the pension rights. $G_t^{P,a} = 1$ implies no real growth, and can therefore be interpreted as inflation indexing. $G_t^{P,a} = G$ implies that pension rights grow at the same rate as wage income. Household income is again given by

$$\begin{aligned} y_t^a = \phi_t^a & \left[\delta_t^a (1 - \tau_t^{W,a}) w_t \ell_t^a \theta_t^a + (1 - \delta_t^a) b_t^a \right] \\ & + (1 - \phi_t^a) \left(1 - \tau_t^{W,i,a}\right) \left[\varsigma_t^a P_t^a + P_t^0 \right] - \tau_t^{l,a}, \end{aligned} \tag{6.4.2}$$

where the only change compared to before is the total gross pension payout. P_t^0 is an age-independent flat pension which is exogenously given. The earnings related part is $\varsigma_t^a P_t^a$ where ς_t^a is simply a policy parameter. In the calibration this parameter is set to 1 but it can be used to model discretionary pension cuts or raises. The household problem can now be written as

$$V(A_t^a, P_t^a) = \max_{C_t^a, \ell_t^a} \quad \left[{}^1\!/\!_\rho \left(Q_t^a\right)^\rho + \beta \gamma_{t+1}^a G^\rho V_{t+1}^{a+1} \right], \qquad \text{s.t.} \tag{6.4.3}$$

$$\text{(6.1.4), (6.4.2) and} \tag{6.4.4}$$

$$Q_t^a = C_t^a - \Psi_t^a, \quad \Psi_t^a = \phi_t^a \left[\delta_t^a \varphi^a \left(\ell_t^a\right) - (1-\delta_t^a) h_t^{a,e} \right].$$

Define the change in remaining life utility at time t to a marginal increase in financial wealth and pension wealth as $\lambda_t^a \equiv \partial V_t^a / \partial A_t^a$ and $\eta_t^a \equiv \partial V_t^a / \partial P_t^a$. The two optimality and the two envelope conditions are

$$C_t^a: \quad (Q_t^a)^{\rho-1} = \gamma_{t+1}^a \beta G^{\rho-1} R_{t+1} \lambda_{t+1}^{a+1} pc_t \tag{6.4.5}$$

$$\ell_t^a: \quad (Q_t^a)^{\rho-1} \phi_t^a \delta_t^a \varphi^{a\prime}(\ell_t^a) = \gamma_{t+1}^a \beta G^{\rho-1} R_{t+1} \lambda_{t+1}^{a+1} \phi_t^a \delta_t^a (1-\tau_t^{W,a}) w_t \theta_t^a + \gamma_{t+1}^a \beta G^{\rho-1} G_t^{P,a} \eta_{t+1}^{a+1} m_t^a \phi_t^a \delta_t^a w_t \theta_t^a \tag{6.4.6}$$

$$A_t^a: \quad \lambda_t^a = \gamma_{t+1}^a \beta G^{\rho-1} R_{t+1} \lambda_{t+1}^{a+1} \tag{6.4.7}$$

$$P_t^a: \quad \eta_t^a = \gamma_{t+1}^a \beta G^{\rho-1} \left[G_t^{P,a} \eta_{t+1}^{a+1} + R_{t+1} \lambda_{t+1}^{a+1} (1-\tau_t^{W,i,a}) \varsigma_t^a (1-\phi_t^a) \right] \tag{6.4.8}$$

The conditions imply the same Euler equation (6.1.12) as before. In contrast - by dividing (6.4.6) by (6.4.5) - the labor supply condition now includes the tax-benefit link implied by the pension system.

$$pc_t \varphi^{a\prime}(\ell_t^a) = (1-\tau_t^{W,a}) w_t \theta_t^a + m_t^a \frac{G_t^{P,a}}{R_{t+1}} \frac{\eta_{t+1}^{a+1}}{\lambda_{t+1}^{a+1}} w_t \theta_t^a. \tag{6.4.9}$$

This can be rewritten as

$$\varphi^{a\prime}(\ell_t^a) = (1-\hat{\tau}_t^{W,a}) w_t \theta_t^a, \tag{6.4.10}$$

with an effective tax rate of

$$\hat{\tau}_t^{W,a} = \frac{\tau_t^C + \tau_t^{W,a} - m_t^a \frac{G_t^{P,a}}{R_{t+1}} \frac{\eta_{t+1}^{a+1}}{\lambda_{t+1}^{a+1}}}{1+\tau_t^C}. \tag{6.4.11}$$

Clearly, the higher m_t^a the stronger the tax-benefit link, the lower the effective tax rate, i.e. with a strong earnings-link the contributions to the pension system are less perceived as taxes. Similarly, the participation decision is changed. The relevant cut-off for the participation rate $\delta_t^a = F^a(\underline{h}_t^a)$ is

$$\underline{h}_t^a = \left[(1-\tau_t^{W,a})w_t\ell_t^a\theta_t^a - b_t^a + m_t^a\frac{G_t^{P,a}}{R_{t+1}}\frac{\eta_{t+1}^{a+1}}{\lambda_{t+1}^{a+1}}w_t\ell_t^a\theta_t^a\right]/pc_t - \varphi^a(\ell_t^a). \quad (6.4.12)$$

The participation tax then changes to

$$\hat{\tau}_t^{\delta,a} = \frac{\tau_t^C + \tau_t^{W,a} + b_t^a/(w_t\ell_t^a\theta_t^a) - m_t^a\frac{G_t^{P,a}}{R_{t+1}}\frac{\eta_{t+1}^{a+1}}{\lambda_{t+1}^{a+1}}}{1+\tau_t^C}. \quad (6.4.13)$$

Define the relative shadow price $\tilde{\eta}_t^a = \eta_t^a/\lambda_t^a$, which after dividing (6.4.8) and (6.4.7) can forward lookingly be expressed as

$$\tilde{\eta}_t^a = (1-\tau_t^{W,i,a})\varsigma_t^a(1-\phi_t^a) + \frac{G_t^{P,a}}{R_{t+1}}\tilde{\eta}_{t+1}^{a+1}. \quad (6.4.14)$$

Treat $\tilde{\eta}$ in the implementation just as another foresight variable. This extension is implemented in `AuerbachKotlikoff_earnings_link`.

6.4.2 Progressive Income Tax Schedule

The idea of modeling progressive taxation was discussed in the intermezzo in section 4.8. Let us briefly discuss the concrete implementation of progressive wage income taxation in the Auerbach-Kotlikoff model with earnings-related pensions. We assume that the income tax burden is given as $T_t^{W,i,a}(x)$ from some income x. This implies the following effective average and marginal burden rates (including social security contributions) for workers and retirees with respective incomes $w_t\theta_t^a\ell_t^a$ and $\varsigma_t^a P_t^a + P_t^0$:

$$\begin{aligned}
\text{average rate worker:} \quad \tau_t^{W,a} &= \frac{T_t^{W,i,a}\left(w_t\theta_t^a\ell_t^a\cdot(1-\nu\tau_t^{W,c})\right)}{w_t\theta_t^a\ell_t^a} + \tau^{W,c} \\
\text{average rate retiree:} \quad \tau_t^{P,a} &= \frac{T_t^{W,i,a}\left(\varsigma_t^a P_t^a + P_t^0\right)}{\varsigma_t^a P_t^a + P_t^0} \\
\text{marginal rate worker:} \quad \tilde{\tau}_t^{W,a} &= T_t^{W,i,a\prime}\left(w_t\theta_t^a\ell_t^a\cdot(1-\nu\tau_t^{W,c})\right) + \tau^{W,c} \\
\text{marginal rate retiree:} \quad \tilde{\tau}_t^{P,a} &= T_t^{W,i,a\prime}\left(\varsigma_t^a P_t^a + P_t^0\right)
\end{aligned}$$

This changes household income to

$$y_t^a = \phi_t^a \left[\delta_t^a(1-\tau_t^{W,a})w_t\ell_t^a\theta_t^a + (1-\delta_t^a)b_t^a\right] \tag{6.4.15}$$

$$+ (1-\phi_t^a)(1-\tau^{P,a})\left[\varsigma_t^a P_t^a + P_t^0\right] - \tau_t^{l,a}, \tag{6.4.16}$$

using the definitions from above. The first order conditions for participation δ_t^a and consumption C_t^a are unaltered. However, the first order condition for hours worked changes and implies an effective tax rate in (6.4.10) of

$$\hat{\tau}_t^{W,a} = \frac{\tau_t^C + \tilde{\tau}_t^{W,a} - m_t^a \frac{G_t^{P,a}}{R_{t+1}} \frac{\eta_{t+1}^{a+1}}{\lambda_{t+1}^{a+1}}}{1+\tau_t^C}. \tag{6.4.17}$$

This implies that (6.4.10) becomes an implicit function of ℓ_t^a which has to be solved numerically for every age group. For numerical reasons it might be convenient to approximate the marginal tax rate function locally, i.e. for the calibrated wage income of a specific representative household.[8] A similar increase of numerical complexity does not occur for the progressive taxation of pensions as they are predetermined by the pension stock from last period, i.e. P_t^a in the algorithm is known in t. The marginal tax rate for retirees appears in the envelope condition for P and consequently in the updated version of (6.4.14) as

$$\tilde{\eta}_t^a = (1-\tilde{\tau}_t^{P,a})\varsigma_t^a(1-\phi_t^a) + \frac{G_t^{P,a}}{R_{t+1}}\tilde{\eta}_{t+1}^{a+1}. \tag{6.4.18}$$

6.4.3 Finance-Constrained Households

Because of perfect foresight households' consumption behavior will follow the permanent income hypothesis, e.g. windfall gains are perfectly smoothed over the whole remaining life-cycle or future income increases have similarly strong effects on today's consumption as on consumption in the period when the income increase actually occurs. Card et al. (2007) shows that empirical consumption behavior is somewhere between predictions of the permanent income hypothesis and hand-to-mouth consumption. Hence, an important ingredient to explain the empirical consumption behavior are finance-

[8] In case of a local point approximation the FOC naturally becomes explicitly solvable again.

constraints of households. An easy way of introducing this to the model is to assume that a share of households is completely finance-constrained. A simple approximative implementation is to assume that this share of households simply consumes whatever their current per-period income is, i.e. $A_t^{C,a} = 0, \ \forall a, t$. The labor market decisions are unaltered compared to the no-income-effect derivations from before. The aggregate consumption effect is a mixture of the reaction of the unconstrained households and the households whose consumption reaction is fully driven by changes in income. The relative mass of those two household types $\{C, U\}$ is then calibrated to replicate the empirical consumption behavior.

6.4.4 Migration

An easy way to introduce migration to the model is to assume exogenous migration shocks. A further simplifying but convenient assumption is that immigration and emigration occurs vis-a-vis an identical country (even in terms of policy shocks). This way one does not have to explicitly model the migration shock in the household decision. The demographic structure is simply

$$N_{t+1}^0 = NB_{t+1} \tag{6.4.19}$$

$$N_{t+1}^{a+1} = \gamma_{t+1}^a N_t^a + Mig_{t+1}^{a+1}, \quad \gamma_t^A = 0 \ \forall t, \tag{6.4.20}$$

where Mig is net immigration. As countries are identical arriving migrants in age-group a bring the same assets A^a and pensions claims P^a as possessed by domestic households. Concerning the pension claims the foreign government is assumed to transfer the corresponding present value of already earned future pension payments to the domestic government. The transfers of assets and intergovernmental pension claims have to be taken into account in the current account. Other than that net migration does not directly change the household problems but only affects the relative weight of the different representative households. Another important consequence of our assumptions is that foreign and domestic workers are indistinguishable and perfect substitutes in the production process.

If one wants to relax this assumption one can explicitly distinguish between domestic and foreign workers, who then can enter production in a more complex way (e.g. not perfectly substitutable) or have different labor market characteristics (e.g. different productivity profiles). This however implies that one has to keep track of two populations, i.e. households differ by age and nationality, i.e. $N_t^{a,n}$ which doubles the number of household problems to be solved.

6.4.5 Skill Choice

Very much like in the extension before one can partition the population, this time in order to reflect different skill or education groups. For simplicity assume that the skill decision occurs at the beginning of life and cannot be altered afterwards. Assuming S different skill groups denote the skill index as $s \in \{1, 2, ..., S\}$, where 1 is the lowest and S is highest attainable skill class. Hence, the demographic laws are

$$N_{t+1}^{0,s} = NB_{t+1}^{s} \tag{6.4.21}$$

$$N_{t+1}^{a+1,s} = \gamma_{t+1}^{a,s} N_t^{a,s}, \quad \gamma_t^{A,s} = 0 \ \forall t, \tag{6.4.22}$$

i.e. there are $S \times (A+1)$ representative household problems to be solved. The decision of which skill level to choose at the beginning of life can be endogenized the following way. Assume that households have different (inverse) learning ability v which is distributed according to cdf $\Gamma(\cdot)$. The assumption is that the lower v the easier it is for a household to learn. Households compare the indirect utility at the beginning of life $V^{0,s}$ to decide which skill to choose. In addition there are disutility costs $c^s(v)$ related to obtaining a skill level fulfilling the following assumptions

$$\frac{\partial c^s(v)}{\partial v} > \frac{\partial c^{s-1}(v)}{\partial v} > 0 \ \ \forall s \in \{2, 3, ..S\} \quad \text{and} \quad c^s(0) = 0 \ \ \forall s, \tag{6.4.23}$$

which implies that it becomes increasingly painful for low ability persons to obtain a higher skill class. Skill choice behavior is determined by the cut-off abilities $\underline{v}$

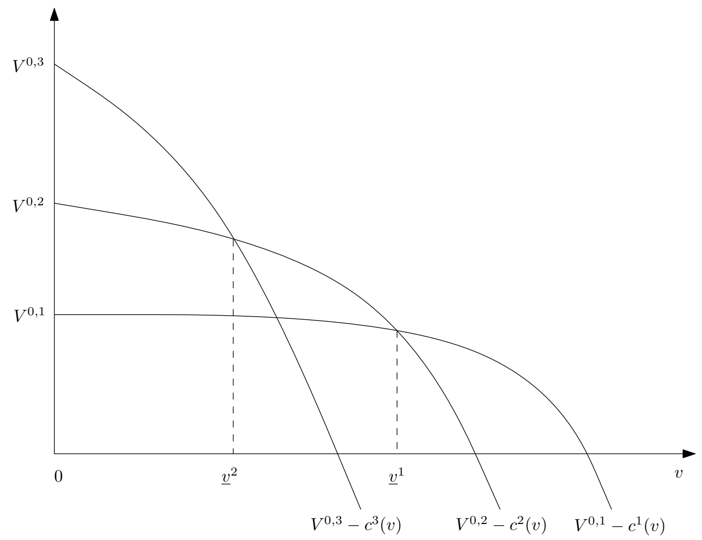
$$V_t^{0,s} - c^s(\underline{v}_t^{s-1}) = V_t^{0,s-1} = c^{s-1}(\underline{v}_t^{s-1}) \ \ \forall s \in \{2, 3, ..S\}, \tag{6.4.24}$$

i.e. whenever $v < \underline{v}_t^{s-1}$ a household will choose a skill of least s. The mass of newborns in every skill group is then given by

$$N_t^{0,s} = N_t^0 \times \begin{cases} 1 - \Gamma\left(\underline{v}_t^1\right) & : s = 1, \\ \Gamma\left(\underline{v}_t^{s+1}\right) - \Gamma\left(\underline{v}_t^s\right) & : s \in \{2, 3, ..., S-1\}, \\ \Gamma\left(\underline{v}_t^S\right) & : s = S. \end{cases} \tag{6.4.25}$$

The decision in the case of $S = 3$ is exemplarily illustrated in the following figure. The households will depending on their ability v always choose a

Figure 6.4.1: The skill choice problem



position on the upper envelope of the three intersecting lines. Households with $0 < v < \underline{v}^2$ choose the highest skill level 3, and so on. The fact that education also has economic costs reflected in the opportunity costs because people have to enter the labor market at a later point can easily be implemented by restricting labor supply exogenously during times of education. Those costs are then already incorporated in the indirect utilities $V^{0,s}$.

6.4.6 Endogenous Retirement

In the Auerbach-Kotlikoff-model choosing endogenous retirement is another discrete choice. Many countries offer options to retire early which is often connected with a reduction in pension payments (while persons can increase their pensions by postponing retirement). Similarly, to the education decision households compare indirect utilities of different options. Assume that households have a window of opportunity between ages $\underline{a}$ and $\bar{a}$ to choose their retirement age. At every $\underline{a} \leq a < \bar{a}$ households have to compare $\bar{a} - a$ different indirect utilities which implies that $\bar{a} - a$ hypothetical remaining life-cycle household problems from a to A have to be solved in addition. Especially when one also wants to allow for a finer choice grid (e.g. choose the month of retirement) the computational complexity increases considerably.

6.4.7 Multi-Country Models

A simple way to model a multi-country link is by assuming perfect goods and capital market integration between a countable number of countries of non-degenerate sizes (following Buiter, 1981). Hence, there is a single real interest rate r applicable in all countries and all countries have an influence on it. The rate is determined in a single asset market. Let us use a front superscript ${}^{j}X$ for $j \in 1, ..., J$ for J countries. The rate r is the clearing price such that $\sum_{j=1}^{J} {}^{j}\zeta^{Y} = 0$.[9] Although comparably slow a rather robust way of solving this kind of multi-country model is to start with a single guess of r and solving the small open economy versions of all the countries sequentially. From the resulting excess demands one can compute an updated guess for r (doing a static adjustment and leaving behavior constant) and so on until convergence. For smaller models (e.g. everything considered in this script) it should be no problem to solve the economies simultaneously. A multi-country setting of the described form can be especially appropriate to study reforms or shocks that occur in many countries at the same time. For example, studying demographic change for a single country using the small open economy assumption with a constant interest rate ignores the fact that population aging is a synchronized trend that affects virtually all (developed) economies and might therefore be grave enough to change the world interest rate.

6.4.8 Income Effects

The Auerbach-Kotlikoff-model presented above differs from the original one proposed in Auerbach and Kotlikoff (1987) in some aspects. One is that the original version is based on an utility representation that gives rise to income

[9]The small open economy case is when ${}^{j}A/(\sum_{j=0}^{J} {}^{j}A) \rightarrow 0$.

effects. Adding income effects to the presented model is shortly explained in this section. For this purpose we just use one labor supply margin (i.e. only choice of hours, no participation or retirement decision). In contrast to the models above the important difference is that households will react to lump-sum gains or deductions (unrelated to the amount of labor they provide) not only by changing their life-time consumption path but also by adjusting their hours supply. The main difference is that the consumption-leisure bundle Q is not linearly separable anymore. Assume that the utility function of a household of age a in recursive form is given as

$$\begin{aligned} V(A_t^a) =& \max_{C_t^a, l_t^a} \; {}^1\!/\!_\rho \left(Q_t^a\right)^\rho + \beta G^\rho \gamma_{t+1}^a V_{t+1}^{a+1} && (6.4.26) \\ & \text{subject to} \\ Q_t^a =& \left[\left(C_t^a\right)^{1-1/\vartheta} + \alpha \delta_t^a \left(l_t^a\right)^{1-1/\vartheta} \right]^{1/(1-1/\vartheta)}, && (6.4.27) \\ G A_{t+1}^{a+1} =& R_{t+1} \left[A_t^a + \bar{y}_t^a - pc_t C_t^a \right], \quad \bar{y}_t^a = y_t^a + iv_t^a + ab_t^a, && (6.4.28) \\ y_t^a =& \delta_t^a (1-\tau_t^{W,a}) w_t \ell_t^a \theta_t^a + (1-\delta_t^a) b_t^a - \tau_t^{l,a}. && (6.4.29) \end{aligned}$$

where σ is again the intertemporal elasticity of substitution and the parameter ϑ denotes the elasticity of substitution between consumption and leisure in the utility CES-aggregate. Recall that $\rho = (\sigma - 1)/\sigma$. l_t^a is leisure, i.e. time endowment E_t^a minus hours supply ℓ_t^a.[10] The choice of l_t^a has to be restricted such that $\ell_t^a \geq 0$ or equivalently $l_t^a \leq E_t^a$. For the moment we assume that the inequality is never binding. Participation rate δ_t^a is assumed to be exogenously given. Define the change in remaining life utility at time t to a marginal increase in financial wealth as $\lambda_t^a \equiv \partial V_t^a / \partial A_t^a$. The two optimality and the envelope conditions are

$$\begin{aligned} C_t^a:&\quad \left(C_t^a\right)^{-1/\vartheta} = \gamma_{t+1}^a \beta G^{\rho-1} R_{t+1} \lambda_{t+1}^{a+1} \left(\Theta_t^a\right)^{-1} \cdot pc_t && (6.4.30) \\ l_t^a:&\quad \left(l_t^a\right)^{-1/\vartheta} = \gamma_{t+1}^a \beta G^{\rho-1} R_{t+1} \lambda_{t+1}^{a+1} \left(\Theta_t^a\right)^{-1} \cdot (1-\tau_t^{W,a}) w_t \theta_t^a / \alpha && (6.4.31) \\ A_t^a:&\quad \lambda_t^a = \gamma_{t+1}^a \beta G^{\rho-1} R_{t+1} \lambda_{t+1}^{a+1} && (6.4.32) \end{aligned}$$

using $\Theta_t^a \equiv \left[\left(C_t^a\right)^{1-1/\vartheta} + \alpha \delta_t^a \left(l_t^a\right)^{1-1/\vartheta} \right]^{\frac{1/\vartheta - 1/\sigma}{1-1/\vartheta}}$. Observe that the Θ-term would vanish in case of $\sigma = \vartheta$. Combining (6.4.30) and (6.4.31) gives a

[10] For detrending the model appropriately with constant labor supply at balanced growth one has to assume that time endowment grows with g as well.

relationship of consumption to leisure

$$l_t^a = (\tilde{w}_t^a/(pc_t\alpha))^{-\vartheta} \cdot C_t^a, \qquad \tilde{w}_t^a \equiv (1-\tau_t^{W,a})w_t\theta_t^a. \tag{6.4.33}$$

This function captures the income effect of labor supply. A windfall gain in assets or non-labor related per-period income will lead to higher wealth and therefore to higher consumption levels, which by (6.4.33) increases leisure and discourages work. Use (6.4.33) to eliminate l_t^a in Θ_t^a which is now

$$\Theta_t^a = (C_t^a)^{(1/\vartheta-1/\sigma)} \cdot \Lambda_t^a, \qquad \Lambda_t^a \equiv \left[1 + (\tilde{w}_t^a/pc_t)^{1-\vartheta}\,\alpha^{\vartheta}\delta_t^a\right]^{\frac{1/\vartheta-1/\sigma}{1-1/\vartheta}} \tag{6.4.34}$$

Insert this in (6.4.30) in combination with (6.4.32) to get

$$(C_t^a)^{-1/\sigma}\Lambda_t^a/pc_t = \lambda_t^a. \tag{6.4.35}$$

Using this expression and its time shifted version again in (6.4.32) gives us the Euler equation

$$GC_{t+1}^{a+1} = \left[\gamma_{t+1}^a \beta R_{t+1} \frac{pc_t}{pc_{t+1}} \frac{\Lambda_{t+1}^{a+1}}{\Lambda_t^a}\right]^{\sigma} \cdot C_t^a. \tag{6.4.36}$$

This expression looks very familiar. The important difference is that in case of $\vartheta \neq \sigma$, Λ becomes a function of the effective wage $\tilde{w}$ and so does the entire Euler equation. To arrive at the consumption function follow the following steps.

First, define $\xi_t^a = (\tilde{w}_t^a/(pc_t\alpha))^{-\vartheta}$, such that $l_t^a = \xi_t^a \cdot C_t^a$. This way we write per period income as

$$\bar{y}_t^a = \tilde{y}_t^a - \delta_t^a\tilde{w}_t^a\xi_t^a \cdot C_t^a, \qquad \tilde{y}_t^a = \delta_t^a\tilde{w}_t^a E_t^a + (1-\delta_t^a)b_t^a + iv_t^a + ab_t^a - \tau_t^{l,a}. \tag{6.4.37}$$

Define the following

$$\tilde{H}_t^a = \tilde{y}_t^a + \frac{G\tilde{H}_{t+1}^{a+1}}{R_{t+1}}, \qquad \hat{H}_t^a = -\delta_t^a\tilde{w}_t^a\xi_t^a \cdot C_t^a + \frac{G\hat{H}_{t+1}^{a+1}}{R_{t+1}}, \tag{6.4.38}$$

such that $H_t^a = \tilde{H}_t^a + \hat{H}_t^a$ and repeat the proof from above slightly altered.

First, rewrite the budget constraint (6.4.28) similar to the chapters before as

$$\begin{aligned} A_t^a &= pc_t C_t^a - \tilde{y}_t^a + \delta_t^a \tilde{w}_t^a \xi_t^a \cdot C_t^a \\ &+ \sum_{s=1}^{\infty} pc_{t+s} C_{t+s}^{a+s} - \tilde{y}_{t+s}^{a+s} + \delta_{t+s}^{a+s} \tilde{w}_{t+s}^{a+s} \xi_{t+s}^{a+s} \cdot C_t^a \prod_{u=t+1}^{t+s} G(R_u)^{-1} \end{aligned} \tag{6.4.39}$$

Define $\phi_t^a = pc_t + \delta_t^a \tilde{w}_t^a \xi_t^a$. Collect terms to express welfare in terms of $\phi \cdot C$.

$$\mathcal{W}_t^a = A_t^a + \tilde{H}_t^a, \qquad \text{where} \tag{6.4.40}$$

$$\mathcal{W}_t^a = \phi_t^a C_t^a + \sum_{s=1}^{\infty} \phi_{t+s}^{a+s} C_{t+s}^{a+s} \prod_{u=t+1}^{t+s} G(R_u)^{-1}, \tag{6.4.41}$$

$$\tilde{H}_t^a = \tilde{y}_t^a + \sum_{s=1}^{\infty} \tilde{y}_{t+s}^{a+s} \prod_{u=t+1}^{t+s} G(R_u)^{-1}. \tag{6.4.42}$$

Insert the Euler equation (6.4.36) in (6.4.41) to arrive at

$$\mathcal{W}_t^a = \phi_t^a C_t^a \Omega_t^a \quad \Rightarrow \quad C_t^a = (\phi_t^a \Omega_t^a)^{-1} \left[A_t^a + \tilde{H}_t^a \right], \tag{6.4.43}$$

$$\Gamma_t^a = (\Lambda_t^a / pc_t)^{\sigma} \phi_t^a + \left(\gamma_{t+1}^a \beta\right)^{\sigma} (R_{t+1})^{\sigma-1} \cdot \Gamma_{t+s}^{a+s}, \tag{6.4.44}$$

$$\Omega_t^a = \Gamma_t^a (\Lambda_t^a / pc_t)^{-\sigma} (\phi_t^a)^{-1}, \tag{6.4.45}$$

$$\tilde{H}_t^a = \tilde{y}_t^a + \frac{G \tilde{H}_{t+1}^{a+1}}{R_{t+1}}. \tag{6.4.46}$$

Once C_t^a is known one can simply compute l_t^a and ℓ_t^a.

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