General Equilbrium Policy Analysis

Lecture 2

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Outline

Solution to Ex. 2

The **marginal utility** for the given isoelastic felicity is

$$u(C) = \frac{\sigma}{\sigma - 1}\left(C^{\frac{\sigma-1}{\sigma}} - 1\right) \qquad \Rightarrow \qquad u'(C) = C^{-\frac{1}{\sigma}}.$$

Insert in (3.1.6) to get the **Euler equation** $C_{t+1} = (\beta R_{t+1})^{\sigma} C_t$ which is inserted in (3.1.9)

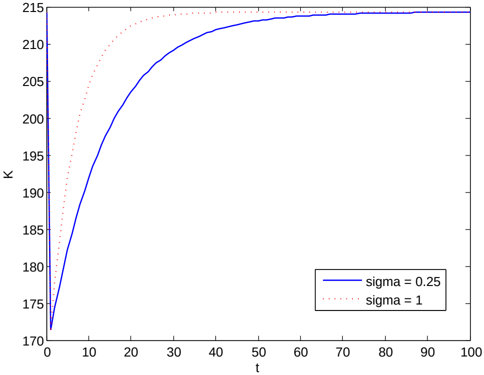
$$\mathcal{W}_t = C_t + \frac{1}{R_{t+1}}\Big[C_t\beta^{\sigma} R^{\sigma}_{t+1} + \frac{1}{R_{t+2}}\Big[C_t\beta^{2\sigma} R^{\sigma}_{t+1} R^{\sigma}_{t+2} + \dots$$

$$\mathcal{W}_t = C_t \cdot \Big[1 + \beta^{\sigma}(R_{t+1})^{\sigma-1} + \beta^{2\sigma}(R_{t+1})^{\sigma-1}(R_{t+2})^{\sigma-1} + \dots$$

$$\mathcal{W}_t = C_t \cdot \Omega_t, \quad \text{where} \quad \Omega_t = 1 + \beta^{\sigma}(R_{t+1})^{\sigma-1}\Omega_{t+1}$$

Ω becomes an **additional foresight variable**, next to H and V.

Solution to Ex. 3

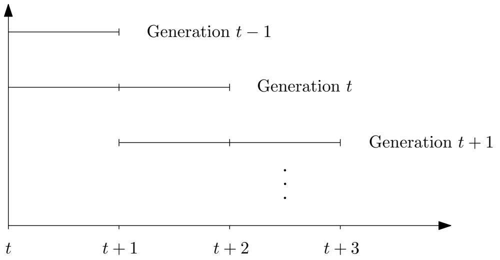


The intertemporal elasticity of substitution σ has strong effects for the '**recovery speed**'. The higher the elasticity, the higher the willingness to shift and postpone consumption, the higher the capital accumulation.

Outline

Diamond Samuelson Model

Figure: Cohort life spans Diamond Samuelson model

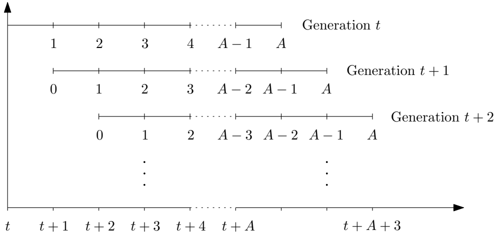


$$N^O_{t+1} = N^Y_t,$$
$$N^Y_{t+1} = NB_{t+1},$$

Age: twice a period.

Auerbach-Kotlikoff Model

Figure: Cohort life spans Auerbach-Kotlikoff model

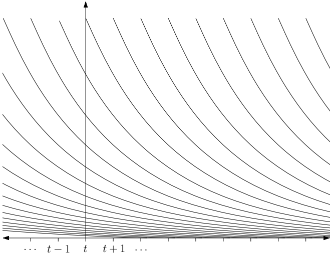


$$N_{t+1}^{a+1} = N_t^a, \ 0 \leq a < A$$
$$N_{t+1}^0 = NB_{t+1}.$$

Age: A times a period (usually a year).

Blanchard Model

Figure: Cohort size Blanchard model



Constant survival rate: $\gamma \Rightarrow$ Age is stochastic. Mass of households born in $v < t$ at time t: $N_{v,t}$.

$$N_{v,t+1} = \gamma N_{v,t}, \; \forall v \leq t$$
$$N_{t+1,t+1} = NB_{t+1}.$$

Life expectancy: $1/(1-\gamma)-1$.

Auerbach-Kotlikoff Model with Mortaltiy

An **Auerbach-Kotlikoff Model** with an **age dependent survival rate** γ^a like the Blanchard model leads to **stochastic life spans**.

$$N_{t+1}^{a+1} = \gamma^a N_t^a,\ 0 \leq a < A,\ \text{with } \gamma^A = 0,$$
$$N_{t+1}^0 = NB_{t+1}$$

Life expectancy: $\sum_{a=1}^{A} a(1-\gamma^a)\prod_{s=0}^{a-1}\gamma^s$.

This specification will be used in **lecture 6**.

Probabilistic Aging

In principle a further generalization is the **Probabilistic Aging** framework (see Grafenhofer et al., 2007) where moving from one age class to the next works **probabilistically** (with prob. $1 - \omega^a$).

This way the models period length can be **detached** from the (average) duration of being in an age class.

$$N_{t+1}^{a+1} = \gamma^{a+1}\omega^{a+1}N_t^{a+1} + \gamma^a(1-\omega^a)N_t^a, \; 1 \leq a < A, \text{ with } \omega^A = 1,$$
$$N_{t+1}^1 = \gamma^1\omega^1 N_t^1 + NB_{t+1}.$$

The **Gertler**-model is a special case with $A = 2$, $\gamma^1 = 1$, $\gamma^2, \omega^1 < 1$.

These specifications will be used in **lecture 5**.

Outline

Assumptions

Entering the realm of overlapping generations models we start with a simple Blanchard model with the following **assumptions**

- Single country, **small open** economy.
- Firms face hd1 **capital-adjustment costs**.
- Isoelastic felicity.
- Exogenous labor supply (ℓ_0 per household).
- A mass of households each facing a constant probability of death $1 - \gamma$.
- Households of same age **insure** themselves against the **risk of longevity**.
- 3 markets: homogenous good (numéraire), labor, and assets.

Demography

Two subscripts: t is the current period, $v \leq t$ is the period of birth of a specific household.

Two assumptions for existence of a final steady state:

- **Survival rate** at the end of period t is γ_{t+1}. The sequence of survival rates fulfills $\lim_{t\to\infty} \gamma_t = \gamma$.
- The **number of newborns** at the end of period t is NB_{t+1}. The sequence of newborns fulfills $\lim_{t\to\infty} NB_t = NB$.

The mass of **households born** at v evolves according to

$$N_{v,t+1} = \gamma_{t+1} N_{v,t}, \quad \forall v \leq t$$
$$N_{t+1,t+1} = NB_{t+1}$$

The mass of **all households** ($N_t = \sum_{v=-\infty}^{t} N_{v,t}$) evolves according to

$$N_{t+1} = \gamma_{t+1} N_t + NB_{t+1}$$

Household problem 1/2

$$U(A_{v,t}) = \max_{C_{v,t}} \; u(C_{v,t}) + \beta\gamma_{t+1}U(A_{v,t+1}), \quad \text{s.t.}$$

$$\gamma_{t+1}A_{v,t+1} = R_{t+1}\left[A_{v,t} + w_{v,t}\ell_0 - C_{v,t}\right]$$

Fair life insurance: End of period assets of dead households $A^{end}_{v,t}(1-\gamma_{t+1})$ are collected and have to be equal to the premium $(\vartheta A^{end}_{v,t})$ for surviving households $\vartheta A^{end}_{v,t}\gamma_{t+1}$, hence $\vartheta = (1-\gamma_{t+1})/\gamma_{t+1}$. Therefore next periods assets are:

$$A_{v,t+1} = (1+\vartheta)A^{end}_{v,t} = A^{end}_{v,t}/\gamma_{t+1}.$$

Similar **optimality** and **envelope** conditions to Ramsey model:

$$C_{v,t}: \quad u'(C_{v,t}) = \beta\lambda_{v,t+1}R_{t+1}$$

$$A_{v,t}: \quad \lambda_{v,t} = \beta\lambda_{v,t+1}R_{t+1}$$

Hence, the **Euler equation** (for iso-elastic felicity) is

$$u'(C_{v,t}) = \beta R_{t+1}u'(C_{v,t+1}) \quad \Rightarrow \quad C_{v,t+1} = (\beta R_{t+1})^{\sigma}C_{v,t}.$$

Household problem 2/2

Recursively insert to rewrite the **budget constraint** as

$$A_{v,t} = C_{v,t} - w_t\ell_0 + \frac{\gamma_{t+1}}{R_{t+1}}\Big[C_{v,t+1} - w_{t+1}\ell_0 + \frac{\gamma_{t+2}}{R_{t+2}}\Big[C_{v,t+2} - w_{t+2}\ell_0 + \dots$$

$$\mathcal{W}_{v,t} = C_{v,t} + \frac{\gamma_{t+1}}{R_{t+1}}\Big[C_{v,t+1} + \frac{\gamma_{t+2}}{R_{t+2}}\Big[C_{v,t+2} + \dots$$

$$H_{v,t} = w_t\ell_0 + \frac{\gamma_{t+1}}{R_{t+1}}\Big[w_{t+1}\ell_0 + \frac{\gamma_{t+2}}{R_{t+2}}\Big[w_{t+2}\ell_0 + \dots = w_t\ell_0 + \frac{\gamma_{t+1}H_{t+1}}{R_{t+1}}$$

Insert the **Euler equation** in the definition of $\mathcal{W}_t$

$$\mathcal{W}_{v,t} = C_{v,t} + \frac{\gamma_{t+1}}{R_{t+1}}\Big[C_{v,t}\beta^{\sigma}R^{\sigma}_{t+1} + \frac{\gamma_{t+2}}{R_{t+2}}\Big[C_{v,t}\beta^{2\sigma}R^{\sigma}_{t+1}R^{\sigma}_{t+2} + \dots$$

$$\mathcal{W}_{v,t} = C_{v,t} \cdot \Big[1 + \beta^{\sigma}\gamma_{t+1}(R_{t+1})^{\sigma-1} + \beta^{2\sigma}\gamma_{t+1}\gamma_{t+2}(R_{t+1}R_{t+2})^{\sigma-1} + \dots$$

$$\mathcal{W}_{v,t} = C_{v,t} \cdot \Omega_t, \quad \text{where} \quad \Omega_t = 1 + \beta^{\sigma}(R_{t+1})^{\sigma-1}\gamma_{t+1}\Omega_{t+1}$$

Hence, the **consumption function** is

$$C_{v,t} = \Omega^{-1}_{v,t}(A_{v,t} + H_{v,t}).$$

Aggregation 1/2

The beauty of the Blanchard model is that it can be **analytically aggregated**, i.e. we can analytically derive characterizations of aggregate consumption, assets, etc. Define the **aggregation rule** for some variable X as

$$X_t = \sum_{v=-\infty}^{t} X_{v,t} N_{v,t}. \tag{3.1}$$

For constant variables and **static relations** aggregation is simple, e.g. $L_0 = \sum_{s=t}^{-\infty} \ell_0 N_s = \ell_0 \cdot N_t$ or

$$C_{v,t} = \Omega_t^{-1}(A_{v,t} + H_{v,t})$$
$$C_{v,t} N_{v,t} = \Omega_t^{-1}(A_{v,t} N_{v,t} + H_{v,t} N_{v,t})$$
$$\sum_{v=-\infty}^{t} C_{v,t} N_{v,t} = \sum_{v=-\infty}^{t} \Omega_t^{-1}(A_{v,t} N_{v,t} + H_{v,t} N_{v,t})$$
$$C_t = \Omega_t^{-1}(A_t + H_t)$$

Aggregation 2/2

Aggregation is more complicated for **dynamic relations**, e.g.

$$H_{v,t} = w_t \ell_0 + \gamma_{t+1} \frac{H_{v,t+1}}{R_{t+1}} \quad \Rightarrow$$

$$\sum_{v=-\infty}^{t} H_{v,t} N_{v,t} = w_t \sum_{v=-\infty}^{t} \ell_0 N_{v,t} + \gamma_{t+1} \frac{\sum_{v=-\infty}^{t} H_{v,t+1} N_{v,t}}{R_{t+1}} \quad \Rightarrow$$

$$H_t = w_t L_0 + \gamma_{t+1} \frac{H_{t+1}(N_t/N_{t+1})}{R_{t+1}}$$

or

$$\gamma_{t+1} A_{v,t+1} = R_{t+1} \left[A_{v,t} + w_t \ell_0 - C_{v,t} \right] \quad \Rightarrow$$

$$\gamma_{t+1} \sum_{v=-\infty}^{t} A_{v,t+1} N_{v,t} = R_{t+1} \left[\sum_{v=-\infty}^{t} \left(A_{v,t} + w_t \ell_0 - C_{v,t} \right) N_{v,t} \right] \Rightarrow$$

$$A_{t+1} = R_{t+1} \left[A_t + w_t L_0 - C_t \right]$$

See detailed proofs in manuscript.

Firm problem 1/2

Maximize **firm value** (present value profits V):

$$
\begin{aligned}
V(K_t) &= \max_{I_t, L_t^D} \; \chi_t + \frac{V(K_{t+1})}{R_{t+1}}, \quad \text{s.t.} \\
\chi_t &= Y_t - w_t L_t^D - I_t - J_t && \text{(per-period profits = dividend)} \\
Y_t &= f(K_t, L_t^D) && \text{(hd1 production function)} \\
K_{t+1} &= (1-\delta^K)K_t + I_t && \text{(capital law of motion)} \\
J_t &= \psi/2 \cdot K_t(I_t/K_t - \delta^K)^2 && \text{(capital adjustment costs)}
\end{aligned}
$$

Define the shadow price of capital as $q_t = V'(K_t)$ then the two **optimality** and the **envelope** conditions are:

$$
\begin{aligned}
I_t : &\quad q_{t+1} = R_{t+1}(1 + J_{I_t}) \\
L_t^D : &\quad Y_{L_t^D} = w_t \\
K_t : &\quad q_t = Y_{K_t} - J_{K_t} + \frac{q_{t+1}}{R_{t+1}}(1-\delta^K)
\end{aligned}
$$

Firm problem 2/2

Theorem

Hayashi's theorem. Firm value and capital stock are related through Tobin's-q

$$q_t K_t = V_t, \quad \forall t$$

Proof

See manuscript.

Combine Hayashi's theorem with the law of motion and the optimality condition for investment to get a **relationship of investments and future profits** in form of a **quadratic function**

$$I_t = \frac{V_{t+1}}{R_{t+1}(1 + J_{I_t})} - (1 - \delta^K)K_t$$

Foreign Assets and the Trade Balance

In a small open economy the **interest rate is fixed**. Instead excess supply of assets is simply '**absorbed**' abroad by the **net foreign assets** position D_t^F. Households can invest in the domestic firm or abroad, i.e.

$$A_t = V_t + D_t^F$$

As foreign asset and good flows are two sides of the same coin

$$D_{t+1}^F = R_{t+1}\left[D_t^F + TB_t\right]$$

where the **trade balance** TB, i.e. net exports, is defined as

$$TB_t = Y_t - C_t - I_t - J_t.$$

Temporary Equilibrium

In period t we know

- all parameters
- **predetermined** variables: K_t and D_t^F.
- guesses for **forward looking** variables: H_{t+1}, V_{t+1} and Ω_{t+1}.

Three markets have to clear: Labor ($L_t^D = L_0$), assets ($A_t = V_t + D_t^F$) and goods ($Y_t = C_t + I_t + J_t + TB_t$).

1. $L_t^D = L_0 \quad \Rightarrow \quad w_t$
2. $I_t = \frac{V_{t+1}}{R_{t+1}(1+J_{I_t})} - (1-\delta^K)K_t \quad \Rightarrow \quad I_t, V_t, K_{t+1}$
3. $H_t = w_t L_0 + \gamma_{t+1} \frac{H_{t+1}(N_t/N_{t+1})}{R_{t+1}}$ and $A_t = V_t + D_t^F \quad \Rightarrow \quad C_t$
4. $TB_t = Y_t - C_t - I_t - J_t \quad \Rightarrow \quad D_{t+1}^F$

Respective code is `Blanchard_demo`.

Outline

Implementation on the PC

Foresight variables: H, V and Ω, **Predetermined**: K and D^F

- We discuss code implementations: `Blanchard_demo`.
- We look at the following
  1. role of capital-adjustment costs
  2. evolution of foreign assets
  3. a positive shock to the number of newborns (Ex. 4)
  4. a positive shock to the survival rate (Ex. 5)

  and interpret the results.

Home exercises due before lecture 3: Ex. 1.