Technical Note:

A Medium-Scale New Keynesian Model

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1 Introduction

This note presents a medium sized New Keynesian model with the following characteristics.

- Either small open economy with exchange rate peg or closed economy
- Finitely lived households (Blanchard, 1985) and constant population size (with infinitely lived households as limiting case)
- Ricardian and rule-of-thumb household types
- CES production
- Capital adjustment costs
- Variable capacity utilization
- Habit persistence in consumption and labor supply
- Government (various taxes and subsidies, government consumption, government investment, seigniorage)

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- Monopolistic value added production with sticky price setting (Calvo assumptions)
- Fixed costs of value added production
- Labor varieties giving rise to market power of the households
- Joint wage setting by unions for every labor variety subject to wage stickiness (Calvo assumptions) following Erceg et al. (2000) and Ratto et al. (2009)
- Competitive final good production
- Competitive capital good production
- Competitive labor packers
- Monetary authorities who peg the exchange rate against the rest of the world (small open economy) or either choose money supply or nominal interest rate via a Taylor rule (closed economy)
- Stationary technology and price level
- Optional multi-industry extension

This note introduces a medium-scale New Keynesian model similar to Eric Sims' lecture notes ('A New Keynesian Model with Price Stickiness' and 'A Medium-Scale New Keynesian DSGE Model') and Galí (2015)'s textbook. However, aggregate uncertainty is ignored; agents are gifted with perfect foresight (see e.g. Hall, 2009), which does not require the usual approximation techniques needed for stochastic models and in addition allows us to analyze permanent shocks.

Notational differences to 'On fiscal multipliers in New Keynesian small open economy models' manuscript:

- Π_t includes (excludes) profit taxes in this document (in the paper)
- current account is TB (CA) in this document (in the paper)
- in multi-sector extension the domestic price level is P^h (P) in this document (in the paper)

2 Nominal versus real values

Domestic money $\mathcal{M}_t$ is used as the unit of valuation, i.e. as the numeraire which means that the price of one unit of money is 1. The price level of the domestic final good is P_t, such that real balances are $\mathcal{M}_t/P_t$. The nominal interest rate is i_{t+1} which means that 1 unit of money invested in the single financial instrument in period t is rewarded with $(1+i_{t+1})$ units of money in period $t+1$. In contrast, just holding 1 unit of money in t gives 1 unit of money in $t+1$. The real interest factor is defined as $R_{t+1} = (1+r_{t+1}) = (1+i_{t+1})P_t/P_{t+1}$, alternatively written as $R_{t+1} = (1+i_{t+1})/(1+\pi_{t+1})$ with the inflation rate defined as $\pi_t = P_t/P_{t-1} - 1$. Note that $r_{t+1} \approx i_{t+1} - \pi_{t+1}$ is only approximately true.

Observe the mapping between dynamic equations expressed in nominal and real terms. Assume some security that promises a nominal payment of X_t at the beginning of every period. Then the discounted nominal value of said security is

$$V_t = X_t + \frac{V_{t+1}}{1+i_{t+1}} \tag{2.1}$$

To express this in real terms use $\hat{V}_t = V_t/P_t$. Divide (2.1) by P_t to get

$$\hat{V}_t = \frac{X_t}{P_t} + \frac{V_{t+1}}{(1+i_{t+1})P_t} = \frac{X_t}{P_t} + \frac{\hat{V}_{t+1}P_{t+1}}{(1+i_{t+1})P_t} = \frac{X_t}{P_t} + \frac{\hat{V}_{t+1}(1+\pi_{t+1})}{(1+i_{t+1})} \tag{2.2}$$

$$\hat{V}_t = \frac{X_t}{P_t} + \frac{\hat{V}_{t+1}}{R_{t+1}} \tag{2.3}$$

In a perfect foresight setting the two ways are equivalent for the choice maker, i.e. setting up optimization problems in real versus nominal terms gives the same (real) results. The subtle difference in case of uncertainty is that i_{t+1} is usually assumed to be known in t while r_{t+1} is not.

3 Labor packers

The continuum of households indexed $l \in [0,1]$ supplies differentiated labor input to a labor packer, who aggregate the individual inputs to a homogeneous labor input for production. The labor packer assembles labor subject to the following CES aggregator

$$\hat{L}_t = \left[\int_0^1 (L_{l,t})^{\frac{\epsilon^w - 1}{\epsilon^w}} dl \right]^{\frac{\epsilon^w}{\epsilon^w - 1}} \tag{3.1}$$

with $\epsilon^w > 1$. Cost minimization gives demand for labor variety l as

$$L_{l,t} = \left(\frac{W_t}{W_{l,t}} \right)^{\epsilon^w} \hat{L}_t, \quad \text{with} \quad W_t = \left[\int_0^1 (W_{l,t})^{1-\epsilon^w} dl \right]^{1/(1-\epsilon^w)} \tag{3.2}$$

as the wage index. The sum of individual labor supply is denoted L_t which is

$$L_t = \int_0^1 L_{l,t} dl = \int_0^1 \left(\frac{W_t}{W_{l,t}} \right)^{\epsilon^w} \hat{L}_t dl. \tag{3.3}$$

Hence, the index of price dispersion drives a wedge between the sum of individually supplied labor and the aggregate homogeneous labor input used in production

$$L_t = v_t^w \hat{L}_t, \qquad v_t^w = \int_0^1 \left(\frac{W_t}{W_{l,t}} \right)^{\epsilon^w} dl, \tag{3.4}$$

where $v_t^w \geq 1$ is the index of wage dispersion. If the index is larger than 1 then sticky wages lead to an output loss, as $\hat{L}_t = L_t / v_t^w$ enters the production function.

4 Households

We assume two types of households – Ricardian that can save, indexed U for unconstrained, and rule-of-thumb that cannot save, indexed C for constrained. Of the total mass $N_t = 1, \ \forall t$ of households an exogenous share π are assumed to be unconstrained and $1 - \pi$ is the mass of constrained households. Households are finitely lived in the spirit of Blanchard (1985) and face a constant mortality rate $1 - \gamma$ every period. Let $N_{v,t}$ denote the mass of households (of both saving types) born at v at time t. With the assumption of constant overall population the evolution of cohorts is described by

$$N_{v,t+1} = \gamma N_{v,t}, \ \forall v \leq t, \tag{4.1}$$

$$N_{t+1,t+1} = (1 - \gamma) N_t. \tag{4.2}$$

Total population is the sum over all cohorts, i.e. $N_t = \sum_{v=-\infty}^{t} N_{v,t}$ which is partitioned in $N_t^U = \pi N_t$ and $N_t^C = (1-\pi)N_t$. The case of infinitely lived households is nested as $\gamma = 1$. All households differ by the labor variety they supply (index by l) which are equally distributed over both saving types and age cohorts.

4.1 Ricardian households

A Ricardian household born in period v and of labor type l faces the following problem

$$U_{v,l,t}^U = \max_{C_{v,t}^U} \frac{(\tilde{C}_{v,t}^U)^{1-\sigma^C} - 1}{1-\sigma^C} - \psi \frac{(\tilde{L}_{l,t})^{1+\sigma^L}}{1+\sigma^L} + \Theta \ln\left(\frac{\mathcal{M}_t^U}{P_t}\right) + \beta\gamma U_{v,l,t+1}^U, \quad (4.3)$$

subject to the following budget constraint where $A_{v,t}^U$ denotes all financial assets that earn after-tax nominal interest i^W,

$$\gamma A_{v,t+1}^U = (1+i_{t+1}^W)\left[A_{v,t}^U + W_t^W \hat{L}_t - P_t^C C_{v,t}^U - P_t \tau_t^L - \Delta\mathcal{M}_t^U\right], \quad (4.4)$$

with $\Delta\mathcal{M}_t^U = \mathcal{M}_t^U - \mathcal{M}_{t-1}^U$, and after-tax prices $i_{t+1}^W = i_{t+1}(1-\tau_t^R)$, $W_t^W = (1-\tau_t^W)W_t$ and $P_t^C = (1+\tau_t^C)\bar{P}_t^C$, where $\bar{P}_t^C$ is the before-tax price level of the consumption basket. The optimization w.r.t. labor supply and money demand is done collectively, i.e. delegated to a trade union and described in the next section. The union distributes average money holdings and average wage income back to the household. This is why v and l indices have already been omitted in the problem set-up where possible. Ex-ante wage risks which stem from the fact that not all wages are reset every period are perfectly shared through said union. Therefore individual ex-ante labor income $(1-\tau_t^W)W_{l,t}L_{l,t}$ is replaced in the budget constraint by average labor income $(1-\tau_t^W)W_t\hat{L}_t$ irrespective of l (and v). This implies that also consumption is independent of l, i.e. $C_{v,l,t}^U = C_{v,k\neq l,t}^U = C_{v,t}^U$. The same is true for assets, which is why already in the problem set-up the labor variety indices were omitted for C and A^U. The risk of uncertain life-expectancy is insured by the usual Blanchard-type reverse life insurance. Note that while labor income is the same for every cohort, asset holdings and consequently consumption in principle differ by cohort, which is why v cannot be dropped for A^U and C.

A household receives interest income on per-period savings consisting of the stock of financial assets plus nominal labor income minus outlays for consumption, lump-sum taxes[1] and the change in money holding. Money is the numeraire. Preferences are defined over $\tilde{C}^U_{v,t} = C^U_{v,t} - \kappa\bar{C}^U_{v,t-1}$ and $\tilde{L}_{l,t} = L_{l,t} - \kappa^L \bar{L}_{t-1}$, where κ and κ^L measure the strength of habit persistence in consumption and labor supply and the bars refer to average consumption and labor supply.[2] $\sigma = 1/\sigma^C$ is the intertemporal elasticity of substitution and $1/\sigma^L$ is the Frisch elasticity of labor supply. Households delegate their wage setting and money demand decisions to a trade union as described in the subsection section and therefore take wages and labor supply as given. The envelope condition and the FOC for consumption are

$$A^U_{v,t}: \quad \lambda_{v,t} = \beta\lambda_{v,t+1}(1 + i^W_{t+1}), \tag{4.5}$$

$$C^U_{v,t}: \quad (\tilde{C}^U_{v,t})^{-\sigma^C} = \beta\lambda_{v,t+1}(1 + i^W_{t+1})P^C_t. \tag{4.6}$$

Combining the envelope condition and the FOC for consumption gives the usual Euler equation

$$(\tilde{C}^U_{v,t})^{-\sigma^C} = \beta(1 + i^W_{t+1})\frac{P^C_t}{P^C_{t+1}}(\tilde{C}^U_{v,t+1})^{-\sigma^C} \quad \Rightarrow \quad \tilde{C}^U_{v,t+1} = \left[\beta R^W_{t+1}\right]^\sigma \tilde{C}^U_{v,t}, \tag{4.7}$$

where $R^W_{t+1} = (1 + i^W_{t+1})\frac{P^C_{t+1}}{P^C_t}$ is the effective real interest factor for the household after interest and consumption taxes and $\sigma = 1/\sigma^C$ is the intertemporal elasticity of substitution. The steps for deriving the consumption function are along the lines described in detail in Schuster (2018) such that the optimal consumption choice of a Ricardian household born in period v can be

[1] The government sets lump-sum taxes measured in terms of the final output good, i.e. outlays on lump-sum taxes (receipts of lump-sum transfers) are price indexed.

[2] This means that the households (and the trade union) do not take into account the consequences of moving the consumption anchor and labor supply anchor for the next period. Ex-post we have $\bar{C}^U_{v,t} = C^U_{v,t}$ and $\bar{L}_t = L_t$.

summarized in the following block

$$
\begin{aligned}
C_{v,t}^{U} &= (\Omega_t P_t^C)^{-1}\left[A_{v,t}^{U} + H_{v,t} - \Delta_t \kappa C_{v,t-1}^{U}\right] + \kappa C_{v,t-1}^{U}, && (4.8)\\
H_{v,t} &= W_t^W \hat{L}_t - P_t \tau_t^L - \Delta \mathcal{M}_t^U + \gamma H_{v,t+1}/(1+i_{t+1}^W), && (4.9)\\
\Lambda_t &= \Delta_t (P_t^C)^{-\sigma} + \beta^\sigma (1+i_{t+1}^W)^{\sigma-1}\gamma \Lambda_{t+1} && (4.10)\\
\Omega_t &= \Lambda_t (P_t^C)^{\sigma-1}, && (4.11)\\
\Delta_t &= P_t^C + \kappa\gamma/(1+i_{t+1}^W)\Delta_{t+1}, && (4.12)\\
\tilde{C}_{v,t}^{U} &= C_{v,t}^{U} - \kappa C_{v,t-1}^{U}. && (4.13)
\end{aligned}
$$

Aggregation over v is described in section 9.

4.2 Rule-of-thumb households

Rule-of-thumb households are assumed to share the same separable utility functions in consumption and labor as the Ricardian households. However, by the existence of financial frictions and/or myopic behavior they do not save and simply consume their disposable income every period. This implies that they do not hold assets and money. Consequently, consumption per rule-of-thumb household does not depend on when the household was born and we omit v. The consumption function is then simply given by

$$
C_t^C = \left[(1-\tau_t^W) W_t \hat{L}_t - P_t \tau_t^L - P_t \tau_t^{LC}\right]/P_t^C, \qquad (4.14)
$$

where τ_t^{LC} is a lump-sum tax (transfer) that is only applied to rule-of-thumb households.

5 Trade union

5.1 Wage setting

Wage setting is carried out by a trade union that exerts the market power given by the existence of differentiated demand for the labor varieties by the labor packers. The downward sloping labor demand curves (3.2) are taken as given by the union when setting a wage for every labor variety l. Wage setting is however limited as a result of wage stickiness of Calvo form. Every period the wage can be reset only for a random share $1-\theta^w$ of households.

Note that the union sets the wage for all households (Ricardian and rule-of-thumb) which means that wage rates can differ by l but not by savings-type or time of birth v. Further, as the mass of type l workers stays constant as a dead worker of type l is immediately replaced by a newborn of type l and death does not trigger an extraordinary wage reset, the union's problem is independent of the survival probability γ. The union faces the following wage setting problem for each labor variety l

$$\max_{W_{l,t}} V^w_{l,s} = W_{l,t} L_{l,s} \tilde{\lambda}_s - \psi \frac{(\tilde{L}_{l,s})^{1+\sigma^L}}{1+\sigma^L} + \theta^w \beta V^w_{l,s+1}, \tag{5.1}$$

starting from $s = t$, where in an abuse of notation $L_{l,s} = L_s(W_{l,t})$ refers to the demand function (3.2) in period s given the individual wage rate fixed in period t. The union uses the marginal utility of average consumption ($\tilde{C}_t = \pi \tilde{C}^U_t + (1-\pi)\tilde{C}^C_t$) over all households in their objective function.[3]

$$\tilde{\lambda}_t = (1-\tau^W_t)(\tilde{C}_t)^{-\sigma^C}/P^C_t. \tag{5.2}$$

Consequently, $L_{l,t}$ as well as $W_{l,t}$ do not differ by savings-type. Note that the while $\tilde{L}_{l,t} \neq L_{l,t}$ in case of $\kappa^L > 0$ we have that

$$\frac{\partial \tilde{L}_{l,t}}{\partial W_{l,t}} = \frac{\partial L_{l,t}}{\partial W_{l,t}} = -\epsilon^w \cdot \frac{L_{l,t}}{W_{l,t}}. \tag{5.3}$$

[3]The weight in the objective function depends on the functional form of the utility function. For example, in case of RINCE-type preferences, e.g. $\left[C_t - \psi \frac{L_t^{1+\sigma^L}}{1+\sigma^L}\right]^{\sigma^C}$ income and disutility of labor are both weighted by the marginal utility of (total) consumption therefore effectively implying that $\tilde{\lambda}_t$ is independent of C_t. Hence, in this case there would be no income effects. An ad-hoc way to eliminate income effects in our specification is to replace $\tilde{C}_t$ by the initial steady state value $\tilde{C}_0$ in (5.2) for all t.

The first order condition is

$$
\begin{aligned}
&\left[\tilde{\lambda}_t(1-\epsilon^w)L_t(W_{l,t}) - \psi\frac{L_t(W_{l,t})}{W_{l,t}}(-\epsilon^w)(\tilde{L}_t(W_{l,t}))^{\sigma^L}\right] + \\
(\beta\theta^w)&\left[\tilde{\lambda}_{t+1}(1-\epsilon^w)L_{t+1}(W_{l,t}) - \psi\frac{L_{t+1}(W_{l,t})}{W_{l,t}}(-\epsilon^w)(\tilde{L}_{t+1}(W_{l,t}))^{\sigma^L}\right] + \\
(\beta\theta^w)^2&\left[\tilde{\lambda}_{t+2}(1-\epsilon^w)L_{t+2}(W_{l,t}) - \psi\frac{L_{t+2}(W_{l,t})}{W_{l,t}}(-\epsilon^w)(\tilde{L}_{t+2}(W_{l,t}))^{\sigma^L}\right] + \\
(\beta\theta^w)^3&\Big[\dots\Big] + \dots = 0.
\end{aligned} \tag{5.4}
$$

Inserting demands gives

$$
\begin{aligned}
&\tilde{\lambda}_t(1-\epsilon^w)W_t^{\epsilon^w}W_{l,t}^{-\epsilon^w}\hat{L}_t - \psi W_t^{\epsilon^w}W_{l,t}^{-\epsilon^w-1}\hat{L}_t(-\epsilon^w)(\tilde{L}_{l,t})^{\sigma^L} + \\
&\beta\theta^w\left[\tilde{\lambda}_{t+1}(1-\epsilon^w)W_{t+1}^{\epsilon^w}W_{l,t}^{-\epsilon^w}\hat{L}_{t+1} - \psi W_{t+1}^{\epsilon^w}W_{l,t}^{-\epsilon^w-1}\hat{L}_{t+1}(-\epsilon^w)(\tilde{L}_{l,t+1})^{\sigma^L}\right] + \\
&(\beta\theta^w)^2\Big[\dots\Big] + \dots = 0.
\end{aligned} \tag{5.5}
$$

Observe that $\tilde{L}_{l,t}$ also depends on $W_{l,t}$ but is taken as given for now. Multiply by $-W_{l,t}^{\epsilon^w+1}/\epsilon^w$ to get

$$
(\epsilon^w-1)/\epsilon^w W_{l,t}\tilde{\lambda}_t W_t^{\epsilon^w}\hat{L}_t - \psi W_t^{\epsilon^w}\hat{L}_t(\tilde{L}_{l,t})^{\sigma^L} + \dots = 0. \tag{5.6}
$$

Now pull out $W_{l,t}$ and rearrange to get an implicit expression for the resetting wage

$$
W_{l,t}^* = \frac{\epsilon^w}{\epsilon^w-1}\frac{\tilde{\Lambda}_t^1}{\tilde{\Lambda}_t^2} \tag{5.7}
$$

$$
\tilde{\Lambda}_t^1 = \psi W_t^{\epsilon^w}\hat{L}_t(\tilde{L}_{l,t})^{\sigma^L} + \theta^w\beta\tilde{\Lambda}_{t+1}^1 \tag{5.8}
$$

$$
\tilde{\Lambda}_t^2 = \tilde{\lambda}_t W_t^{\epsilon^w}\hat{L}_t + \theta^w\beta\tilde{\Lambda}_{t+1}^2 \tag{5.9}
$$

$$
\tilde{L}_{l,t} = (W_t/W_{l,t}^*)^{\epsilon^w}\hat{L}_t - \kappa^L L_{t-1}. \tag{5.10}
$$

Observe that it the same for all households, i.e. $W_{l,t}^* = W_t^*$. As will become apparent later, note that in case of no wage stickiness we will have $W_t = W_t^*$. Assume further that the share of constrained households is zero, i.e. $\pi = 1$. Then the first order condition collapses to

$$
\frac{W_t^W}{P_t^C} = \frac{\epsilon^w}{\epsilon^w-1}\psi(\tilde{L}_t)^{\sigma^L}(\tilde{C}_t)^{\sigma^C} = \frac{\epsilon^w}{\epsilon^w-1}MRS_t, \tag{5.11}
$$

i.e. the after-tax real wage, shortly written as $w_t = W_t^W/P_t^C$, is set the marginal rate of substitution times a constant mark-up factor, $\mu^w = \frac{\epsilon^w}{\epsilon^w - 1} > 1$ that approaches 1 in the limiting case of perfect substitutability of labor varieties, i.e. if $\epsilon^w \to \infty$. To measure the variable size of the wage markup in our model we therefore define $\mu_t^w = w_t/MRS_t$ which in case of wage stickiness will in general differ from $\epsilon^w/(\epsilon^w - 1)$.

5.2 Money demand

Money demand is modeled as a collective decision of the unconstrained households via the union in order to preserve tractability.[4] Mimicing the individual preferences the union optimizes the following objective function by choosing optimal average money demand

$$V_t^{\mathcal{M}} = \max_{\mathcal{M}_t^U} \; \Theta \ln\left(\frac{\mathcal{M}_t^U}{P_t}\right) + \beta V_{t+1}^{\mathcal{M}}, \quad \text{s.t.} \tag{5.12}$$

$$A_{t+1}^{\mathcal{M}} = (1 + i_{t+1}^W)\left[A_t^{\mathcal{M}} - \Delta\mathcal{M}_t^U\right], \text{ with } \Delta\mathcal{M}_t^U = \mathcal{M}_t^U - \mathcal{M}_{t-1}^U. \tag{5.13}$$

The shadow price is $\partial V_t^{\mathcal{M}}/\partial A_t^{\mathcal{M}} \equiv \lambda_t$ which is set to the marginal utility of average consumption of the unconstrained household, i.e. $\lambda_t = (\tilde{C}_t^U)^{\sigma^C}/P^C$. The envelope condition and the FOC are

$$A_t^{\mathcal{M}}: \quad \lambda_t = \beta\lambda_{t+1}(1 + i_{t+1}^W), \tag{5.14}$$

$$\mathcal{M}_t^U: \quad \Theta/\mathcal{M}_t^U - \beta\lambda_{t+1}(1 + i_{t+1}^W) + \beta^2\lambda_{t+2}(1 + i_{t+2}^W) = 0. \tag{5.15}$$

The FOC for money holding can be combined with the envelope conditions for period t and $t+1$ to get

$$\Theta/\mathcal{M}_t^U - \lambda_t + \beta\lambda_{t+1} = 0 \quad \Rightarrow \quad \Theta/\mathcal{M}_t^U - \lambda_t + \lambda_t/(1 + i_{t+1}^W) = 0 \tag{5.16}$$

Where in the last step the envelope condition was inserted once more. Collecting λ_t and inserting its definition gives average money demand per un-

[4] This part is somewhat ad-hoc but only needed to close the money market in case the monetary authorities directly target money supply instead of the interest rate (Taylor rule). Alternatively, one could not microfound money demand at all and simply impose a downward sloping demand curve as often done in comparable models.

constrained household

$$\Theta/\mathcal{M}_t^U = \lambda_t i_{t+1}^W/(1+i_{t+1}^W) \quad \Rightarrow \quad \frac{\mathcal{M}_t^U}{P_t^C} = \frac{1+i_{t+1}^W}{i_{t+1}^W}\Theta(\tilde{C}_t^U)^{\sigma^C}. \tag{5.17}$$

6 Final goods

Production is carried out by a competitive final goods assembling firm subject to the following CES production function

$$Y_t = \left[\int_0^1 (Y_{i,t})^{\frac{\epsilon-1}{\epsilon}} di\right]^{\frac{\epsilon}{\epsilon-1}} \tag{6.1}$$

with $\epsilon > 1$. Cost minimization gives demand for variety i as

$$Y_{i,t} = \left(\frac{P_t}{P_{i,t}}\right)^{\epsilon} Y_t, \quad \text{with } P_t = \left[\int_0^1 (P_{i,t})^{1-\epsilon} di\right]^{1/(1-\epsilon)} \tag{6.2}$$

as the price index which is taken as given by the individual value added good producer.

7 Capital goods

Value added goods production requires labor and capital. Capital is rented from a competitive capital goods producer who builds up the economy wide capital stock by investing (in terms of final goods). The optimization problem is maximizing the nominal value of the capital goods firm w.r.t. investment I_t and capacity utilization o_t

$$V_t^C = \max_{I_t, o_t} \Pi_t^C + V_{t+1}^C/(1+i_{t+1}), \quad \Pi_t^C = P_t^K \hat{K}_t - P_t^I(I_t + J_t) - T^F, \tag{7.1}$$

where $\hat{K}_t = o_t K_t$, subject to the law of motion for capital, capital adjustment costs J_t and net taxes T_t^F

$$K_{t+1} = (1-\delta_t)K_t + I_t, \quad \delta_t = \delta_0 + \delta_1(o_t - 1) + \delta_2/2(o_t-1)^2. \tag{7.2}$$

$$J(I_t, K_t) = \frac{1}{2}\psi K_t \left(\frac{I_t}{K_t} - \delta_0\right)^2, \tag{7.3}$$

$$\text{with}\quad J_{I_t} = \psi\left(\frac{I_t}{K_t} - \delta_0\right), \qquad J_{K_t} = \psi/2\left[\delta_0^2 - \left(\frac{I_t}{K_t}\right)^2\right].$$

$$T_t^F = \tau_t^{prof}\left[P_t^K \hat{K}_t - P_t^I \delta_0 K_t\right] - sub_t^L P_t - sub_t^I P_t^I I_t. \tag{7.4}$$

The envelope and the optimality conditions are

$$I_t : q_{t+1} = (1 + i_{t+1}) P_t^I (1 - sub_t^I + J_{I_t}) \tag{7.5}$$

$$o_t : P_t^K = \frac{q_{t+1}}{(1 + i_{t+1})}\left[\delta_1 + \delta_2(o_t - 1)\right]/(1 - \tau_t^{prof}) \tag{7.6}$$

$$K_t : \quad q_t = (1 - \tau_t^{prof}) P_t^K o_t - P_t^I J_{K_t} + P_t^I \delta_0 \tau_t^{prof} + \frac{q_{t+1}}{(1 + i_{t+1})}(1 - \delta_t). \tag{7.7}$$

Hayashi (1982)'s theorem implies $q_t K_t = V_t^C - V_t^R$, where V_t^R is the discounted sum of all firm rents, in our case simply

$$V_t^R = P_t sub_t^L + \frac{V_{t+1}^R}{1 + i_{t+1}}. \tag{7.8}$$

Combining this with the law of motion and the optimality condition for I_t gives the usual investment function

$$I_t = \frac{V_{t+1}^C - V_{t+1}^R}{(1 + i_{t+1}) P_t^I (1 - sub_t^I + J_{I_t})} - (1 - \delta_t) K_t. \tag{7.9}$$

This results in a quadratic equation in I_t of the following form

$$\begin{aligned}&[\psi/K_t]\, I_t^2 + \left[1 - sub_t^I - \psi\delta_0 + \psi(1 - \delta_t)\right] I_t \\ &+ (1 - \delta_t) K_t (1 - sub_t^I - \psi\delta_0) - \left[V_{t+1}^C - V_{t+1}^R\right] / \left[P_t^I (1 + i_{t+1})\right] = 0.\end{aligned} \tag{7.10}$$

The section closes with a proof of Hayashi (1982)'s theorem

Proof. Take the envelope condition for capital (7.7), multiply both sides by

K_t and expand the right hand side by $\frac{q_{t+1}}{1+i_{t+1}} I_t$.

$$q_t K_t = (1 - \tau_t^{prof}) P_t^K o_t K_t - P_t^I J_{K_t} K_t + P_t^I \tau_t^{prof} \delta_0 K_t \\ - \frac{q_{t+1}}{1+i_{t+1}} I_t + \frac{q_{t+1}}{1+i_{t+1}} \left[(1-\delta_t) K_t + I_t\right], \tag{7.11}$$

$$q_t K_t = (1 - \tau_t^{prof}) P_t^K \hat{K}_t - P_t^I J_{K_t} K_t + P_t^I \tau_t^{prof} \delta_0 K_t \\ - P_t^I (1 - sub_t^I + J_{I_t}) I_t + \frac{q_{t+1}}{1+i_{t+1}} K_{t+1}, \tag{7.12}$$

$$q_t K_t = \Pi_t^C - P_t sub_t^L + \frac{q_{t+1}}{1+i_{t+1}} K_{t+1}. \tag{7.13}$$

From the first to the second equation we used the law of motion (7.2) and the optimality condition for investment (7.5). From the second to the last equation we used Euler's theorem and the linear homogeneity of the adjustment cost function and the definition of per-period profits Π^C. Solving forward yields Hayashi (1982)'s result

$$q_t K_t = \sum_{s=t}^{\infty} \Pi_s^C \prod_{u=t+1}^{s} \frac{1}{1+i_u} - \sum_{s=t}^{\infty} P_s sub_s^L \prod_{u=t+1}^{s} \frac{1}{1+i_u} = V_t^C - V_t^R. \tag{7.14}$$

■

8 Value added goods

There is a fixed number of value added goods producers[5] with mass 1. Value added goods are produced as varieties in monopolistic competition. Producer $i \in [0,1]$ uses the following CES-technology

$$Y_{i,t} = \Phi_t \left[\alpha^{1-\rho} \hat{K}_{i,t}^{\rho} + (1-\alpha)^{1-\rho} \hat{L}_{i,t}^{\rho}\right]^{1/\rho}. \tag{8.1}$$

where Φ_t is total factor productivity which depends on the public capital stock, i.e. $\Phi_t = \Phi(K_t^G)$, and $\sigma^P = 1/(1-\rho)$ is the elasticity of substitution

[5] In comparable models this type of producers is typically referred to as 'intermediate goods producers'. To avoid confusions in case of the optional multi-industry extension (see appendix) where goods in their final form (section 6) can either be used for final demand or as inputs in other industries we refer to the producers in this section rather as 'value added producers'.

between capital and labor.[6] Homogeneous labor and effective capital (including capacity utilization) are rented at the competitive after-tax nominal rates $W_t^F = (1+\tau_t^F)W_t$ and $P_t^F = (1+\tau_t^K)P_t^K$. Firms further face fixed costs (FC_t) in terms of the final good, which are assumed to be small enough to guarantee non-negative profits, which are taxed at rate τ_t^{prof}. All costs are deductible which implies that the profit tax does not distort factor input nor price setting of the variety producers.[7] However, fixed costs will influence the amount of 'pure' profits that are taxed relative to the capital goods firm for which profit taxes are distortive. Per-period profits of producer i are given as

$$\Pi_{i,t} = (1-\tau_t^{prof})\left[P_{i,t}Y_{i,t} - P_t^F\hat{K}_{i,t} - W_t^F\hat{L}_{i,t} - P_tFC_t\right]. \tag{8.2}$$

Cost minimization for given prices and output $Y_{i,t}$ is written as

$$\min_{\hat{K}_{i,t},\hat{L}_{i,t}} W_t^F\hat{L}_{i,t} + P_t^F\hat{K}_{i,t} \quad \text{s.t.} \quad Y_{i,t} = \Phi_t\left[\alpha^{1-\rho}\hat{K}_{i,t}^{\rho} + (1-\alpha)^{1-\rho}\hat{L}_{i,t}^{\rho}\right]^{1/\rho}. \tag{8.3}$$

Combining the first order conditions gives the typical result

$$\left[\frac{W_t^F}{P_t^F}\right]^{\sigma^P} = \frac{1-\alpha}{\alpha}\frac{\hat{K}_{i,t}}{\hat{L}_{i,t}}. \tag{8.4}$$

First, note that given the static nature of the optimal input choice minimizing nominal or real costs gives the same result. Second, optimal capital-labor ratio is the same for all firms, even if $Y_{i,t}$ and therefore $\hat{K}_{i,t}$ and $\hat{L}_{i,t}$ differ. The solution for input demands is

$$\hat{K}_{i,t} = \alpha\left[\frac{MC_t}{P_t^F}\right]^{\sigma^P}\Phi_t^{\sigma^P-1}\cdot Y_{i,t}, \quad \hat{L}_{i,t} = (1-\alpha)\left[\frac{MC_t}{W_t^F}\right]^{\sigma^P}\Phi_t^{\sigma^P-1}\cdot Y_{i,t}, \tag{8.5}$$

with nominal marginal costs for one additional unit of $Y_{i,t}$ of[8]

$$MC_t = \left[\alpha(P_t^F)^{1-\sigma^P} + (1-\alpha)(W_t^F)^{1-\sigma^P}\right]^{1/(1-\sigma^P)}/\Phi_t. \tag{8.6}$$

[6] The limiting case $\sigma^P = 1$ gives the Cobb-Douglas production function $Y_{i,t} = \Phi_t\hat{K}_{i,t}^{\alpha}\hat{L}_{i,t}^{1-\alpha}$, where in calibration Φ absorbs the additional $\alpha^{-\alpha}(1-\alpha)^{\alpha-1}$ term.

[7] Section 11 discusses overall profit taxation in more detail.

[8] In the limiting case $\sigma^P = 1$ (Cobb-Douglas) marginal costs are given by $MC_t = \left(\frac{P_t^F}{\alpha}\right)^{\alpha}\left(\frac{W_t^F}{1-\alpha}\right)^{1-\alpha}/\Phi_t$.

Note that the marginal costs are the same for all firms (we therefore drop i). In the Calvo pricing setting only a random fraction $(1-\theta)$ can reset their prices in a given period t. Hence, the problem of choosing the optimal price becomes dynamic as firms have to take expectations about the future when setting the price today, given that they are on average stuck with the choice for $1/(1-\theta)$ periods. The firm discounts its nominal profits from $t+1$ to t with the nominal interest factor $(1+i_{t+1})$.[9] Then an updating (V^P) and a non-updating (V^{NP}) firm's discounted nominal profits (ignoring fixed costs which will not be part of the FOCs anyway) are given as

$$V_t^P = \max_{P_{i,t}}(1-\tau_t^{prof})\,(P_{i,t}-MC_t)\,Y_{i,t} + \frac{\theta V_{t+1}^{NP}(P_{i,t}) + (1-\theta)V_t^P}{1+i_{t+1}} \tag{8.7}$$

$$V_t^{NP}(P_i) = (1-\tau_t^{prof})\,(P_i-MC_t)\,Y_{i,t} + \frac{\theta V_{t+1}^{NP}(P_i) + (1-\theta)V_t^P}{1+i_{t+1}}. \tag{8.8}$$

Note that future optimizations do not affect the current price choice. Likewise, an updating firm sets its price irrespective of the price of the previous period. Hence, what happens in case of $(1-\theta)$ can be ignored and the problem is reduced to

$$\max_{P_{i,t}} V_s = (1-\tau_s^{prof})\,(P_{i,t}-MC_s)\,Y_{i,s} + \theta\frac{V_{s+1}}{1+i_{s+1}}, \tag{8.9}$$

starting with $s=t$. Note that while all s-indexed variables change with time $P_{i,t}$ stays that same as in the first period t. Further, note the difference in where the max-operator appears. Now insert the demand functions (6.2) to get

$$\max_{P_{i,t}} V_s = (1-\tau_s^{prof})\,(P_{i,t}-MC_s)\left(\frac{P_s}{P_{i,t}}\right)^{\epsilon} Y_s + \theta V_{s+1}/(1+i_{s+1}). \tag{8.10}$$

The first order condition (using $\wedge_{t+1} = 1+i_{t+1}$) is

$$\begin{aligned}
&\left[(1-\epsilon)P_{i,t}^{-\epsilon} + MC_t\epsilon P_{i,t}^{-\epsilon-1}\right] Y_t P_t^{\epsilon} + \\
\theta/\wedge_{t+1}\, &\left[(1-\epsilon)P_{i,t}^{-\epsilon} + MC_{t+1}\epsilon P_{i,t}^{-\epsilon-1}\right] Y_{t+1} P_{t+1}^{\epsilon} + \\
\theta^2/(\wedge_{t+1}\wedge_{t+2})\, &\left[(1-\epsilon)P_{i,t}^{-\epsilon} + MC_{t+2}\epsilon P_{i,t}^{-\epsilon-1}\right] Y_{t+2} P_{t+2}^{\epsilon} + \cdots = 0.
\end{aligned} \tag{8.11}$$

[9] Recall that setting this problem up as discounting real profits by the real interest factor gives the same results.

Note that the profit tax drops out and does not affect the price setting decision. Divide by $\epsilon P_{i,t}^{-\epsilon-1}$ and pull out $P_{i,t}\frac{\epsilon-1}{\epsilon}$ out of the infinite sum to get

$$P_{i,t}\frac{\epsilon-1}{\epsilon}\left[Y_tP_t^{\epsilon}+\theta/\wedge_{t+1}Y_{t+1}P_{t+1}^{\epsilon}+\theta^2/(\wedge_{t+1}\wedge_{t+2})Y_{t+2}P_{t+2}^{\epsilon}\dots\right]=$$
$$\left[MC_tY_tP_t^{\epsilon}+\theta/\wedge_{t+1}MC_{t+1}Y_{t+1}P_{t+1}^{\epsilon}+\theta^2/(\wedge_{t+1}\wedge_{t+2})MC_{t+2}Y_{t+2}P_{t+2}^{\epsilon}\dots\right]$$

Hence, the optimal resetting price in period t is

$$P_t^*=\frac{\epsilon}{\epsilon-1}\frac{\Phi_t^1}{\Phi_t^2} \tag{8.12}$$
$$\Phi_t^1=MC_tY_tP_t^{\epsilon}+\theta\Phi_{t+1}^1/(1+i_{t+1}) \tag{8.13}$$
$$\Phi_t^2=Y_tP_t^{\epsilon}+\theta\Phi_{t+1}^2/(1+i_{t+1}) \tag{8.14}$$

Note that it is the same for all firms, we therefore drop the index i. Define $\mu_t^*=\frac{\epsilon}{\epsilon-1}\frac{\Phi_t^1}{\Phi_t^2}/MC_t$ as the markup of the resetting price such that $P_t^*=\mu_t^*MC_t$. Observe that if $\theta=0$ the solution collapses to $P_t^*=\frac{\epsilon}{\epsilon-1}MC_t$, i.e. the markup is constant $\mu_t^*=\mu^*=\frac{\epsilon}{\epsilon-1}$.

9 Aggregation

Nominal dividends of the value added goods firms distributed to the households are

$$\Pi_t=(1-\tau_t^{prof})\int_0^1\Big(P_{i,t}Y_{i,t}-W_t\hat{L}_{i,t}-P_t^K\hat{K}_{i,t}-P_tFC_t\Big)\,di. \tag{9.1}$$

This can be rewritten using $\hat{L}_t=\int_0^1\hat{L}_{i,t}di$ and $\hat{K}_t=\int_0^1\hat{K}_{i,t}di$ as

$$\frac{\Pi_t}{1-\tau_t^{prof}}=\int_0^1(P_{i,t}Y_{i,t})\,di-W_t^F\hat{L}_t-P_t^F\hat{K}_t-P_tFC_t. \tag{9.2}$$

Inserting demand for $Y_{i,t}=(P_{i,t}/P_t)^{-\epsilon}Y_t$ gives

$$\frac{\Pi_t}{1-\tau_t^{prof}}=Y_tP_t^{\epsilon}\int_0^1\left(P_{i,t}^{1-\epsilon}\right)di-W_t^F\hat{L}_t-P_t^F\hat{K}_t-P_tFC_t. \tag{9.3}$$

Use the definition of the price index (6.2) and divide by $P_t^{1-\epsilon} = \int_0^1 (P_{i,t})^{1-\epsilon} di$ to get

$$\Pi_t = (1 - \tau_t^{prof}) \left[\hat{Y}_t P_t - W_t^F \hat{L}_t - P_t^F \hat{K}_t \right], \tag{9.4}$$

with $\hat{Y}_t = Y_t - FC_t$ and define the total firm values of all value added goods producers as

$$V_t^I = \Pi_t + \frac{V_{t+1}^I}{1 + i_{t+1}}. \tag{9.5}$$

Equating the individual demands for variety i with the corresponding production functions gives

$$Y_{i,t} = (P_{i,t}/P_t)^{-\epsilon} Y_t = \Phi_t \left[\alpha^{1-\rho} \hat{K}_{i,t}^{\rho} + (1-\alpha)^{1-\rho} \hat{L}_{i,t}^{\rho} \right]^{1/\rho}. \tag{9.6}$$

Integrate over all firms to get

$$\int_0^1 Y_{i,t} di = Y_t \int_0^1 (P_{i,t}/P_t)^{-\epsilon} di = \int_0^1 \Phi_t \left[\alpha^{1-\rho} \hat{K}_{i,t}^{\rho} + (1-\alpha)^{1-\rho} \hat{L}_{i,t}^{\rho} \right]^{1/\rho} di. \tag{9.7}$$

Now, rewrite the right hand side as $\Phi_t \left[\alpha^{1-\rho} (\hat{K}_{i,t}/\hat{L}_{i,t})^{\rho} + (1-\alpha)^{1-\rho} \right]^{1/\rho} \hat{L}_{i,t}$ by exploiting linear homogeneity and recall that optimality requires all capital-labor ratios to be equal.

$$\int_0^1 Y_{i,t} di = Y_t \int_0^1 (P_{i,t}/P_t)^{-\epsilon} di = \Phi_t \left[\alpha^{1-\rho} (\hat{K}_t/\hat{L}_t)^{\rho} + (1-\alpha)^{1-\rho} \right]^{1/\rho} \int_0^1 \hat{L}_{i,t} di. \tag{9.8}$$

Hence,

$$Y_t = \frac{\Phi_t \left[\alpha^{1-\rho} \hat{K}_t^{\rho} + (1-\alpha)^{1-\rho} \hat{L}_t^{\rho} \right]^{1/\rho}}{v_t}, \quad \text{with } v_t = \int_0^1 (P_{i,t}/P_t)^{-\epsilon} di \tag{9.9}$$

$v_t \geq 1$ is the index of price dispersion ('Tack Yun'-distortion, see Yun, 1996). If the index is larger than 1 sticky prices lead to an output loss.

Note that the definition of the current price level P_t still depends on all individual prices

$$P_t^{1-\epsilon} = \int_0^1 P_{i,t}^{1-\epsilon} di, \tag{9.10}$$

The Calvo assumptions allow us to aggregate out the heterogeneity by proceeding as follows. A fraction $1-\theta$ of firms resets their current price to $P_{i,t}^*$ while a share θ still charges the same price as last period, hence

$$P_t^{1-\epsilon} = \int_0^{1-\theta} (P_t^*)^{1-\epsilon} di + \int_{1-\theta}^1 P_{i,t-1}^{1-\epsilon} di = (1-\theta)(P_t^*)^{1-\epsilon} + \int_{1-\theta}^1 P_{i,t-1}^{1-\epsilon} di. \quad (9.11)$$

As firms are randomly chosen and because of the law of large numbers the last term can be rewritten as

$$P_t^{1-\epsilon} = (1-\theta)(P_t^*)^{1-\epsilon} + \theta \int_0^1 P_{i,t-1}^{1-\epsilon} di, \quad (9.12)$$

Inserting the definition of the price index of $t-1$ yields the aggregate law of motion for the price level

$$P_t^{1-\epsilon} = (1-\theta)(P_t^*)^{1-\epsilon} + \theta (P_{t-1})^{1-\epsilon}. \quad (9.13)$$

We proceed analogously to rid the price dispersion index v_t of heterogeneity. In addition, expand the last term by $(P_{t-1}/P_{t-1})^{-\epsilon}$

$$v_t = \int_0^1 \left(\frac{P_{i,t}}{P_t}\right)^{-\epsilon} di = (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\epsilon} + \theta \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} \int_0^1 \left(\frac{P_{i,t-1}}{P_{t-1}}\right)^{-\epsilon} di. \quad (9.14)$$

The last term is the definition of v_{t-1} such that the evolution of price dispersion follows the following law

$$v_t = (1-\theta)(P_t^*/P_t)^{-\epsilon} + \theta (P_{t-1}/P_t)^{-\epsilon} v_{t-1}. \quad (9.15)$$

With the aggregate price level defined we can compute the average markup of the production price index as $\mu_t = P_t/MC_t$.

Similarly, the definition of the current wage level W_t still depends on all individual wages

$$W_t^{1-\epsilon^w} = \int_0^1 W_{l,t}^{1-\epsilon^w} dl, \quad (9.16)$$

We proceed exactly like before. The Calvo assumptions allow us to aggregate out the heterogeneity by proceeding as follows. A fraction $1-\theta^w$ of households reset their current wage to W_t^* while a share θ^w still charges the same price as last period, hence

$$W_t^{1-\epsilon^w} = \int_0^{1-\theta^w} (W_t^*)^{1-\epsilon^w} dl + \int_{1-\theta^w}^1 W_{l,t-1}^{1-\epsilon^w} dl$$
$$= (1-\theta^w)(W_t^*)^{1-\epsilon^w} + \int_{1-\theta^w}^1 W_{l,t-1}^{1-\epsilon^w} dl. \tag{9.17}$$

As households are randomly chosen and because of the law of large numbers the last term can be rewritten as

$$W_t^{1-\epsilon^w} = (1-\theta^w)(W_t^*)^{1-\epsilon^w} + \theta^w \int_0^1 W_{l,t-1}^{1-\epsilon^w} dl, \tag{9.18}$$

Inserting the definition of the wage index of $t-1$ yields the aggregate law of motion for the price level

$$W_t^{1-\epsilon^w} = (1-\theta^w)(W_t^*)^{1-\epsilon^w} + \theta^w (W_{t-1})^{1-\epsilon^w}. \tag{9.19}$$

We proceed analogously to rid the wage dispersion index v_t^w of heterogeneity. In addition, expand the last term by $(W_{t-1}/W_{t-1})^{-\epsilon^w}$

$$v_t^w = \int_0^1 \left(\frac{W_{l,t}}{W_t}\right)^{-\epsilon^w} dl$$
$$= (1-\theta^w)\left(\frac{W_t^*}{W_t}\right)^{-\epsilon^w} + \theta^w \left(\frac{W_{t-1}}{W_t}\right)^{-\epsilon^w} \int_0^1 \left(\frac{W_{l,t-1}}{W_{t-1}}\right)^{-\epsilon^w} dl. \tag{9.20}$$

The last term is the definition of v_{t-1}^w such that the evolution of price dispersion follows the following law

$$v_t^w = (1-\theta^w)(W_t^*/W_t)^{-\epsilon^w} + \theta^w (W_{t-1}/W_t)^{-\epsilon^w} v_{t-1}^w. \tag{9.21}$$

We next show that the sum over all individual labor incomes is indeed equal to the average wage income distributed by the union to the households times the number of households 1. We start by integrating over all individual labor incomes

$$\int_0^1 W_{l,t} L_{l,t} dl. \tag{9.22}$$

Inserting the individual labor demand curves (3.2) gives

$$\int_0^1 W_{l,t} L_{l,t} dl = W_t^{\epsilon^w} \hat{L}_t \int_0^1 W_{l,t}^{1-\epsilon^w} dl. \tag{9.23}$$

Now, use (9.16) to arrive at

$$\int_0^1 W_{l,t} L_{l,t} dl = W_t^{\epsilon^w} \hat{L}_t W_t^{1-\epsilon^w}, \tag{9.24}$$

hence,

$$\int_0^1 W_{l,t} L_{l,t} dl = W_t \hat{L}_t. \tag{9.25}$$

Given the mass 1 assumption of households, $W_t \hat{L}_t$ is the aggregate wage sum as well as the average wage income by household.

Next, we have to aggregate the system characterizing consumption over all cohorts of unconstrained households. Define for some variable X the average over all cohorts as $X_t = \left[\sum_{v=-\infty}^{t} X_{v,t} N_{v,t}^U\right] / N_t^U$, with $N_t^U = \pi$. Then using the usual aggregation steps for the Blanchard (1985)-type models we can express the block characterizing average consumption behavior as

$$C_t^U = (\Omega_t P_t^C)^{-1} \left[A_t^U + H_t - \Delta_t \kappa \gamma C_{t-1}^U\right] + \kappa \gamma C_{t-1}^U, \tag{9.26}$$

$$A_{t+1}^U = (1 + i_{t+1}^W) \left[A_t^U + W_t^W \hat{L}_t - P_t^C C_t^U - P_t \tau_t^L - \Delta \mathcal{M}_t^U\right], \tag{9.27}$$

$$H_t = W_t^W \hat{L}_t - P_t \tau_t^L - \Delta \mathcal{M}_t^U + \gamma H_{t+1}/(1 + i_{t+1}^W), \tag{9.28}$$

$$\Lambda_t = \Delta_t (P_t^C)^{-\sigma} + \beta^\sigma (1 + i_{t+1}^W)^{\sigma-1} \gamma \Lambda_{t+1} \tag{9.29}$$

$$\Omega_t = \Lambda_t (P_t^C)^{\sigma-1}, \tag{9.30}$$

$$\Delta_t = P_t^C + \kappa\gamma/(1 + i_{t+1}^W) \Delta_{t+1}, \tag{9.31}$$

$$\tilde{C}_t^U = C_t^U - \kappa \gamma C_{t-1}^U. \tag{9.32}$$

Last, aggregate consumption, aggregate assets and aggregate money demand are given as

$$C_t = \pi C_t^U + (1-\pi) C_t^C, \quad A_t = \pi A_t^U, \quad \mathcal{M}_t = \pi \mathcal{M}_t^U, \tag{9.33}$$

while the law of motion of aggregate assets is

$$A_{t+1} = (1 + i^W_{t+1}) \left[A_t + W^W_t \hat{L}_t - P^C_t C_t - T^L_t - \Delta \mathcal{M}_t \right]. \tag{9.34}$$

To describe the aggregate consumption profile explicitly aggregate the individual Euler equation of the unconstrained households (4.7) to arrive at

$$\tilde{C}^U_{t+1} + \frac{1-\gamma}{\gamma} (\Omega_{t+1} P^C_{t+1})^{-1} \left[A^U_{t+1} - \Delta_{t+1} \kappa \gamma C^U_t \right] = \left[\beta R^W_{t+1} \right]^{\sigma} \tilde{C}^U_t. \tag{9.35}$$

Multiply by π, expand the tilde-terms and insert $\pi C^U_t = C_t - (1-\pi) Y^C_t$ where Y^C is disposable income of the unconstrained households equal to C^C to get the law of motion for aggregate consumption

$$\begin{aligned} & C_{t+1} - \gamma\kappa C_t - (1-\pi) \left[Y^C_{t+1} - \gamma\kappa Y^C_t \right] + \\ & \frac{1-\gamma}{\gamma} (\Omega_{t+1} P^C_{t+1})^{-1} \left[A_{t+1} - \Delta_{t+1} \kappa\gamma (C_t - (1-\pi) Y^C_t) \right] = \\ & \left[\beta R^W_{t+1} \right]^{\sigma} \left[C_t - \gamma\kappa C_{t-1} - (1-\pi) \left[Y^C_t - \gamma\kappa Y^C_{t-1} \right] \right]. \end{aligned} \tag{9.36}$$

Observe how this extended aggregate Euler equation collapses to the simple familiar form for the special case $\kappa = 0$ and $\gamma = \pi = 1$,

$$C_{t+1} = \left[\beta R^W_{t+1} \right]^{\sigma} C_t. \tag{9.37}$$

10 Final demands and foreign trade

Demand for final goods can stem from different sources: private consumption, private investment, public consumption, public investment and foreign sources (in case the economy is open). The (before-tax) price of the domestic final good is P and P^m for the imported final good which are imperfect substitutes. We assume the same class of sub-utility/production functions of CES-form but allow for different parameterization depending on the source of demand. In particular we can set different import shares and different elasticities for substitution between the domestic and imported final good. We assume the same sub-utility parameters independent of savings or labor variety type such that when using linear homogeneous CES-aggregators we can directly split aggregate demand for the composite consumption good C.[10] Sub-utility for consumption is assumed to be of the following CES-form

$$C = \left[(\xi^C)^{1-\varepsilon^C} (C^m)^{\varepsilon^C} + (1-\xi^C)^{1-\varepsilon^C} \left(C^h\right)^{\varepsilon^C} \right]^{1/\varepsilon^C} . \tag{10.1}$$

C^m and C^h are imported and domestically produced quantities. Next, we compute compensated unit demands c^m and c^h and the unit expenditure function by solving

$$P^C = \min_{c^m, c^h} \left\{ P^{C,m} c^m + P^{C,h} c^h \right\} \quad \text{s.t.} \quad C(c^m, c^h) = 1, \tag{10.2}$$

where $P^{C,m} = (1+\tau^C)P^m$ and $P^{C,h} = (1+\tau^C)P$. As solution we get the usual unit expenditure function P^C[11] and unit demand functions for the CES-form

$$P^C = \left[\xi^C \left(P^{C,m}\right)^{1-\lambda^C} + (1-\xi^C) \left(P^{C,h}\right)^{1-\lambda^C} \right]^{\frac{1}{1-\lambda^C}} , \tag{10.3}$$

$$c^m = \xi^C \left[P^C / P^{C,m} \right]^{\lambda^C} , \quad c^h = (1-\xi^C) \left[P^C / P^{C,h} \right]^{\lambda^C} , \tag{10.4}$$

where $\lambda^C = 1/(1-\varepsilon^C)$.[12] The solution of splitting composite consumption is then simply $C^h = c^h \cdot C$ and $C^m = c^m \cdot C$. For the other demands (C^G, I^G, I) we proceed analogously just using different superscripts. Note that we abstract from taxation of the other final good uses, such that $P^{j,h} = P$ and $P^{j,m} = P^m$ for $j \in \left\{C^G, I^G, I\right\}$.[13] With a fixed exchange rate and a constant foreign price P^m export demand, i.e. foreign demand for domestic goods, is assumed of the simple form[14]

$$E = E_0 \, (P)^{-\vartheta} , \tag{10.5}$$

[10]We drop the time index unless required for understanding in this section for convenience because of the static nature of the problems.

[11]As the same tax rate applies to domestic as well as imported consumption goods, the before tax price index is given as $\bar{P}^C = P/(1+\tau^C)$.

[12]In the limiting case of a Cobb-Douglas specification, i.e. $\lambda^C = 1$, the price index is given as $P^C = (P^{C,m})^{\xi^C} (P^{C,h})^{1-\xi^C}$.

[13]By assuming the same quasi-preferences for investment and capital adjustment we pull them together and define $I^h = i^h(I+J)$, etc.

[14]When using many different calibrations leading to different amounts of exports it may be convenient to convert the iso-elastic form into a semi-elastic specification that converts a relative price change into a drop in the export to (calibrated) GDP share in percentage points instead of a relative change in E.

where ϑ measures the responsiveness of export demand to relative changes in the domestic price, i.e. the terms of trade. The trade balance[15] measured in domestic currency is

$$TB_t = P_t E_t - P_t^m \left[C_t^m + C_t^{G,m} + I_t^m + I_t^{G,m} \right], \tag{10.6}$$

which dictates the accumulation of foreign assets

$$D_{t+1}^F = (1 + i_{t+1}) \left[D_t^F + TB_t \right]. \tag{10.7}$$

The outlined specification nests the closed economy case by setting $E_0 = 0$ and $\xi^j = 0$ for $j \in \left\{ C, C^G, I, I^G \right\}$ such that $TB_t = DF_t = 0$ for all t.

Let $VA_t = P_t \hat{Y}_t$ be gross value added. We can then write nominal gross domestic product by the production, the expenditure and the income approach:

$$GDP_t = VA_t + T_t^C, \tag{10.8}$$
$$GDP_t = P_t^C C_t + P^{C^G} C_t^G + P_t^I (I_t + J_t) + P_t^{I^G} I_t^G + TB_t, \tag{10.9}$$
$$GDP_t = W_t^F \hat{L}_t + P_t^F \hat{K}_t + \Pi_t / (1 - \tau_t^{prof}). \tag{10.10}$$

Changes in real GDP are computed by replacing current prices in (10.9) by their initial calibration values.

11 Government

The government faces an intertemporal budget constraint of the following form[16]

$$D_{t+1}^G = (1 + i_{t+1}) \left[D_t^G - PB_t \right], \qquad PB_t = Rev_t - Exp_t, \tag{11.1}$$

[15] Without the need to differentiate between goods and services the terms ‘current account’ and ‘trade balance’ are used synonymously.

[16] In some case it may make sense to additionally define government debt in real terms, i.e. $D_t^G = D_t^{G,real} P_t$ or $D_{t+1}^{G,real} = (1 + r_{t+1}) \left[D_t^{G,real} - PB_t / P_t \right]$.

in equilibrium. Expenditure Exp_t is given by public consumption and investment and transfered subsidies

$$Exp_t = P_t^{C^G} C_t^G + P_t^{I^G} I_t^G + sub_t^L P_t + sub_t^I P_t^I I_t. \tag{11.2}$$

Revenue comes from profit taxation, consumption taxes, lump-sum taxes, wage taxes, pay-roll taxes, capital usage taxes, interest taxes and seignioarage.

$$Rev_t = T_t^{prof} + T_t^C + T_t^L + T_t^W + T_t^K + T_t^R + \Delta\mathcal{M}_t, \tag{11.3}$$

where $T_t^R = \tau_t^R \frac{i_{t+1}}{1+i_{t+1}} S_t$ and per-period saving is $S_t = A_t + W_t^W \hat{L}_t - P_t^C C_t - T_t^L$. Note that the budget constraint $A_{t+1} = (1 + i_{t+1}^W) S_t$ can be rewritten as $A_{t+1} = (1 + i_{t+1}) \left[S_t - T_t^R\right]$. Aggregate profit tax revenue is given as

$$\begin{aligned} T_t^{prof} &= \tau_t^{prof} \left[P_t^K \hat{K}_t - P_t^I \delta_0 K_t\right] + \tau_t^{prof} \left[Y_t P_t - W_t^F \hat{L}_t - P_t^F \hat{K}_t\right] \\ &= \tau_t^{prof} \left[P_t \hat{Y}_t - P_t^I \delta_0 K_t - \tau_t^K P_t^K \hat{K}_t - (1 + \tau_t^F) W_t \hat{L}_t\right]. \end{aligned} \tag{11.4}$$

Other revenue components are defined as follows

$$T_t^C = \tau_t^C \bar{P}_t^C C_t, \tag{11.5}$$

$$T_t^L = P_t \left[\tau_t^L + (1 - \pi) \tau_t^{LC}\right], \tag{11.6}$$

$$T_t^W = (\tau_t^F + \tau_t^W) W_t \hat{L}_t, \tag{11.7}$$

$$T_t^K = \tau_t^K P_t^K \hat{K}_t. \tag{11.8}$$

The public capital stock evolves according to

$$K_{t+1}^G = (1 - \delta^G) K_t^G + I_t^G, \tag{11.9}$$

and influences total factor productivity in the following form

$$\Phi(K_t^G) = A_0 \cdot (K_t^G)^{\sigma^G}. \tag{11.10}$$

12 Monetary authorities

Depending on the assumption concerning the openness of the economy we distinguish two cases: small open economy with exchange rate peg and a closed economy setting with two policy options. Based on the case the monetary authorities carries out different policies.

12.1 Case 1: Exchange rate peg

In the case of a small open economy we assume that the monetary authorities peg the exchange rate against the (homogeneous) rest of the world. The domestic nominal (risk-unadjusted) interest rate $\hat{i}$ is pinned down by the uncovered interest rate parity

$$(1 + \hat{i}_{t+1}) = (1 + i^*_{t+1})\mathcal{E}_{t+1}/\mathcal{E}_t, \tag{12.1}$$

where $\mathcal{E}$ is the nominal exchange rate and i^* the foreign nominal interest rate. Hence, in the case of an exchange rate peg we have $\hat{i}_{t+1} = i^*_{t+1}$. The actual domestic nominal risk-adjusted interest rate i is subject to a risk-premium modeled as being (symmetrically) dependent on the foreign asset position[17] (see e.g. Schmitt-Grohé and Uribe, 2003)

$$i_{t+1} = i^*_{t+1} * \exp\left(-\rho^i \left[D^F_t/VA_t - D^F_0/VA_0\right]\right), \tag{12.2}$$

where the 0-subscript indicates calibration values such that in calibration $i = i^*$. ρ^i measures the sensitivity of the risk-premium to changes in the foreign asset position. In general the foreign interest is taken as constant, i.e. $i^*_{t+1} = i^*$ for all t.

12.2 Case 2a: Monetary authority chooses money supply

If the economy is assumed to be closed the monetary authorities follow one of two options. In the first option we let the monetary authority directly choose nominal money supply which in combination with money demand (5.17) pins down the price level, i.e. $\mathcal{M}^S_t$ is set to a value and inserted in (5.17).

$$\mathcal{M}_t = \mathcal{M}^S_t. \tag{12.3}$$

[17]Foreign asset positions are measured in relation to gross value added instead of GDP because of numerical convenience, which could easily be altered.

12.3 Case 2b: Monetary authority chooses nominal interest rate (Taylor rule)

In the second option we let the monetary authority set the interest rate which implicitly pins down the price level. The monetary authority uses the following Taylor rule

$$i_{t+1} = (1-\rho_i)i + \rho_i i_t + (1-\rho_i)\left[\phi_P \pi_t + \phi_P^0 (P_t/P_0 - 1)\right] + \epsilon_{i_t}. \tag{12.4}$$

where ϵ_{i_t} is a discretionary shock which is 0 in the long run. P_0 is the targeted price level (exogenously chosen, e.g. by normalizing $P_0 = 1$) which prevails in the (deterministic) steady state. ϕ_P measures the reaction to a change in prices versus the previous period (recall that $\pi_t = P_t/P_{t-1} - 1$). The higher ϕ_P the stronger the monetary authority will 'lean against' expansionary fiscal policy. The last term in the bracket is added for the sole purpose of anchoring the price level in steady state. We must have $0 \geq \rho_i < 1$ and $\phi_P^0 > 0$ (though it can be very small) otherwise the price level is undetermined. Money demand (5.17) is not an equilibrium condition anymore but can be used to back out the implied money supply for the chosen nominal interest rate.

13 Steady State

Most of the steady state equations are trivial to derive: we simply have to drop the time index of the static relationships. We therefore report the steady state relationships only of the dynamic equations.

$$C^U = \left[A^U + H\right](\Omega P^C)^{-1} / \left[1 - \kappa\gamma + (\Omega P^C)^{-1}\right], \tag{13.1}$$

$$A^U = \left[P^C C^U + P\tau^L - W^W \hat{L}\right](1 + i^W)/i^W, \tag{13.2}$$

$$A = \left[P^C C + T^L - W^W \hat{L}\right](1 + i^W)/i^W, \tag{13.3}$$

$$H = \left[W^W \hat{L} - P\tau^L\right](1 + i^W)/(i^W + 1 - \gamma), \tag{13.4}$$

$$\Delta = P^C(1 + i^W)/(i^W + 1 - \kappa\gamma), \tag{13.5}$$

$$\Lambda = \left[\Delta(P^C)^{-\sigma}\right] / \left[1 - \beta^\sigma (1 + i^W)^{\sigma-1}\gamma\right], \tag{13.6}$$

$$\tilde{C}^U = (1 - \kappa\gamma)C^U, \quad \tilde{C}^C = (1 - \kappa\gamma)C^C, \tag{13.7}$$

$$V^R = Psub^L(1 + i)/i, \quad V^C = \Pi^C(1 + i)/i, \quad V^I = \Pi(1 + i)/i, \tag{13.8}$$

$$DF = -TB(1 + i)/i, \quad DG = PB(1 + i)/i, \tag{13.9}$$

$$K = I/\delta, \quad K^G = I^G/\delta^G. \tag{13.10}$$

Optimal investment behavior of the capital goods firm is described in steady state by

$$q = (1 + i)P^I(1 - sub^I + J_I) \tag{13.11}$$

$$P^K/P^I = \frac{\left[\delta_1 + \delta_2(o - 1)\right](1 - sub^I + J_I)}{1 - \tau^{prof}} \tag{13.12}$$

$$P^K o/P^I = \frac{(i + \delta)(1 - sub^I + J_I) + J_K - \delta_0\tau^{prof}}{1 - \tau^{prof}}. \tag{13.13}$$

Normalizing $o = 1$ and $J = J_I = J_K = 0$ in steady state therefore implies that we set $\delta_1 = i^{emp} + \delta_0\left[1 - \tau^{prof}/(1 - sub^I)\right]$. Further, $\delta_0 = \delta^{emp}$ and δ^2 is used to gauge the sensitivity of capacity utilization. Hence, if $\delta_2 \to \infty$ we are back in the case of constant depreciation and constant capacity utilization. Equating (13.12) and (13.13) pins down o in steady state. As long as the steady state nominal interest rate, profit tax and investment subsidy are equal to their calibration values we have that $o = 1$ in steady state.[18]

[18] Note that if in steady state any of those changed, we would have $o \neq 1$ which means that our normalization of capital adjustment costs to 0 in steady state does no longer hold, i.e. $J = J_I = J_K \neq 0$.

The steady state version of the law of motion of the price level (9.13) is

$$(1-\theta)P^{1-\epsilon} = (1-\theta)(P^*)^{1-\epsilon}, \quad \text{hence} \quad \Rightarrow P^* = P. \tag{13.14}$$

Combining this result with the steady state version of the law of motion of the price dispersion index (9.15) reveals that $v = 1$.[19] The steady state values of Φ^1 and Φ^2 are

$$\Phi^1 = MCYP^\epsilon/(1-\theta/(1+i)), \qquad \Phi^2 = YP^\epsilon/(1-\theta/(1+i)). \tag{13.15}$$

Therefore the price is

$$P = P^* = \frac{\epsilon}{\epsilon - 1}MC. \tag{13.16}$$

We proceed similarly for the wage level. The steady state version of (9.19) implies $W = W^*$. In combination with (9.21) this implies $v^w = 1$ and therefore $\tilde{L} = (1-\kappa^L)L = (1-\kappa^L)\hat{L}$.[20] The steady state values of $\tilde{\Lambda}^1$ and $\tilde{\Lambda}^2$ are

$$\tilde{\Lambda}^1 = \psi W^{\epsilon^w}\hat{L}(\tilde{L})^{\sigma^L}/(1-\theta^w\beta) \tag{13.17}$$

$$\tilde{\Lambda}^2 = \tilde{\lambda}W^{\epsilon^w}\hat{L}/(1-\theta^w\beta). \tag{13.18}$$

Hence,

$$W = W^* = \frac{\epsilon^w}{\epsilon^w - 1}\frac{\psi(\tilde{L})^{\sigma^L}}{\tilde{\lambda}}, \quad \Rightarrow \quad \frac{W^W}{P^C} = \frac{\epsilon^w}{\epsilon^w - 1}MRS, \tag{13.19}$$

where MRS is the average marginal rate of substitution as used by the trade union.

[19] Note that this is only true in case of assuming a stationary price level. If we assumed a stationary positive inflation everything would have to be expressed in terms of price changes and we'll have $v > 1$ in steady state, i.e. a long-run output loss from price stickiness.

[20] Again, this is only true in absence of positive trend wage inflation.

14 Walras' Law

Define the following excess demands

$$
\begin{aligned}
\zeta_t^Y &= C_t^h + I_t^h + C_t^{G,h} + I_t^{G,h} + E_t - \hat{Y}_t, && (14.1)\\
\zeta_t^A &= V_t^I + V_t^C + D_t^G + D_t^F - A_t, && (14.2)\\
\zeta_t^L &= \hat{L}_t^D - \hat{L}_t^S, && (14.3)\\
\zeta_t^L &= \hat{K}_t^D - \hat{K}_t^S, && (14.4)\\
\zeta_t^G &= Rev_t - Exp_t - PB_t, && (14.5)\\
\zeta_t^{\mathcal{M}} &= \mathcal{M}_t - \mathcal{M}_t^S. && (14.6)
\end{aligned}
$$

We omitted the continuum of excess demands for the varieties of value added goods as they are zero by construction of the model. $\hat{Y} = Y - FC$ is net final goods after deduction of fixed costs. We added superscripts D and S to homogeneous labor $\hat{L}$ and effective capital $\hat{K}$ to indicate demand and supply, while in an abuse of notation they were omitted so far in the description. Start with the aggregated budget constraint of the households

$$\frac{A_{t+1}}{1+i_{t+1}} = \left[A_t + W_t^W \hat{L}_t^S - P_t^C C_t - T_t^L - \mathcal{M}_t + \mathcal{M}_{t-1} - T_t^R\right], \qquad (14.7)$$

and eliminate A using (14.2) and money demand $\mathcal{M}$ using (14.6) to get

$$
\begin{aligned}
\frac{V_{t+1}^I + V_{t+1}^C + D_{t+1}^G + D_{t+1}^F - \zeta_{t+1}^A}{1+i_{t+1}} &= V_t^I + V_t^C + D_t^G + D_t^F - \zeta_t^A + W_t^W \hat{L}_t^S \\
&\quad - P_t^C C_t - T_t^L - \zeta_t^{\mathcal{M}} - \mathcal{M}_t^S + \mathcal{M}_{t-1} - T_t^R. \qquad (14.8)
\end{aligned}
$$

Now insert the definitions of V_t^I and V_t^C as well as (14.2) to get

$$
\begin{aligned}
\frac{-\zeta_{t+1}^A + D_{t+1}^G + D_{t+1}^F}{1+i_{t+1}} &= P_t \hat{Y}_t - W_t^F \hat{L}_t^D - P_t^F \hat{K}_t^D - T_t^{prof} + P_t sub_t^L + sub_t^I P_t^I I_t \\
&\quad + P_t^K \hat{K}_t^S - P_t^I (I_t + J_t) + D_t^G + D_t^F - \zeta_t^A + W_t^W \hat{L}_t^S \\
&\quad - P_t^C C_t - T_t^L - \zeta_t^{\mathcal{M}} - \mathcal{M}_t^S + \mathcal{M}_{t-1} - T_t^R. \qquad (14.9)
\end{aligned}
$$

Next, collect all revenue items and replace them by $Rev_t = \zeta_t^G + P_t^{C^G} C_t^G + P_t^{I^G} I_t^G + P_t sub_t^L + sub_t^I P_t^I I_t + PB_t$ to get

$$\frac{-\zeta_{t+1}^A + D_{t+1}^G + D_{t+1}^F}{1+i_{t+1}} = P_t \hat{Y}_t - W_t \hat{L}_t^D - P_t^K \hat{K}_t^D + P_t^K \hat{K}_t^S - P_t^I (I_t + J_t) + D_t^G \\ + D_t^F - \zeta_t^A \ + W_t \hat{L}_t^S - \bar{P}_t^C C_t - \zeta_t^{\mathcal{M}} \\ - \zeta_t^G - P_t^{C^G} C_t^G - P_t^{I^G} I_t^G - PB_t. \tag{14.10}$$

Now, split all final demands, i.e. $P^I (I_t + J_t) = P_t I_t^h + P_t^m I_t^m$, etc., expand by $P_t E_t$ and insert (14.1) to get

$$\frac{-\zeta_{t+1}^A + D_{t+1}^G + D_{t+1}^F}{1+i_{t+1}} = P_t \zeta_t^Y - W_t \hat{L}_t^D - P_t^K \hat{K}_t^D + P_t^K \hat{K}_t^S - P_t^m I^m + D_t^G \\ + D_t^F - \zeta_t^A \ + W_t \hat{L}_t^S - P_t^m C_t^m - \zeta_t^{\mathcal{M}} \\ - \zeta_t^G - P_t^m C_t^{G,m} - P_t^m I_t^{G,m} - PB_t + P_t E_t. \tag{14.11}$$

Insert (10.6), (10.7), (14.3), (14.4) and the law of motion for government debt (11.1) and rearrange to get Walras' Law

$$P_t \zeta_t^Y + W_t \zeta_t^L + P_t^K \zeta_t^K + \zeta_t^A + \zeta_t^G + \zeta_t^{\mathcal{M}} - \frac{\zeta_{t+1}^A}{1+i_{t+1}} = 0. \tag{14.12}$$

In steady state the condition reads

$$P \zeta^Y + W \zeta^L + P^K \zeta^K + \zeta^G + \zeta^{\mathcal{M}} + \frac{i}{1+i} \zeta^A = 0. \tag{14.13}$$

A A multi-sector version of the model

This section just briefly sketches the required changes needed to extend the model described above to a multi-sector setting that can then be calibrated to the corresponding national input-output tables.[21] The economy is comprised of n discrete sectors or industries. In each industry value added is produced by a mass 1 of monopolistically competitive variety producers each producing $Y_{k,i}$ with $k \in \{1, 2, \ldots, n\}$ and $i \in [0, 1]$. In the first stage of final goods production the value added good varieties within an industry are competitively assembled to Y_k at price P_k. Capital is accumulated industry-specifically (i.e. there are n capital good firms) while labor is assumed to be mobile between sectors such that there is a unique wage rate W. Fixed costs of the variety producers are expressed in terms of the industry specific composite value added good at price P_k. Aggregating over varieties works as before leading to n aggregate laws of motion for prices P_k as result of the Calvo assumption.

The second stage of final good assembly is where the input-output structure comes into play. To assemble final good F_k the composite value added good $\hat{Y}_k = Y_k - FC_k$ is required as well as final goods from other industries. Demand for the other final goods is labeled $M_{j,k}$, i.e. the demand for final good from sector j to be used as intermediate in sector k. The sectoral production function is given as

$$F_k = \min \left\{ \frac{M_{1k}}{a_{1k}}, \frac{M_{2k}}{a_{2k}}, \ldots, \frac{M_{nk}}{a_{nk}}, \frac{\hat{Y}_k}{a_{0k}} \right\}. \tag{A.1}$$

a_{jk} denote the fixed input-output coefficients. The coefficients form the familiar matrix $A = [a_{jk}]$ where k is the column index and j is the row index. Producing one unit of good k therefore requires a_{1k} of good 1, a_{2k} of good 2, a_{kk} of good k itself, etc. and a_{0k} of the sector-specific value-added good. The producer price of F_k is P_k^h. Each intermediate good M_{jk} can be sourced domestically or from abroad (assuming imperfect substitutability). The buyer's price of the input M_{jk} is P_{jk}^M which is a price index composed of $P_{jk}^{M,h}$ and $P_{jk}^{M,m}$. Effective intermediate prices are $P_{jk}^{M,h} = P_j^h(1 + \tau_{jk}^{M,h})$ and

[21] The multi-industry set-up closely follows Keuschnigg and Kohler (1994). All time indicies are dropped in this section for easier readability.

$P_{jk}^{M,m} = P_j^m(1+\tau_{jk}^{M,m})$. The input price indices are given as

$$P_{jk}^M = \min_{m_{jk}^h, m_{jk}^m} \left\{ P_{jk}^{M,h} m_{jk}^h + P_{jk}^{M,m} m_{jk}^m \quad \text{s.t.} \;\; M_{jk}\left(m_{jk}^h, m_{jk}^m\right) \geq 1 \right\}, \quad \text{(A.2)}$$

where M_{jk} is again assumed to be a CES aggregator with share parameter ξ^M and substitution elasticity λ^M. Splitting the input coefficients results in $a_{ji}^h = m_{ji}^h a_{ji}$ and $a_{ji}^m = m_{ji}^m a_{ji}$. Note that while input-output coefficients a_{ji} are constant the split between domestic and imports is price sensitive. The input-output matrix of domestic production is therefore $A^h = \left[a_{jk}^h\right]$. As also the second stage of final goods assembling is done competitively the zero profit condition implies the following relationship between final goods producer price P_k^h, value added composite good price P_k and intermediate good prices $P_{j,k}^M$

$$\sum_j a_{jk} P_{jk}^M + a_{0k} P_k - P_k^h = 0. \quad \text{(A.3)}$$

Final demands aggregation is extended by one additional stage, such that $C = \text{CES}_\text{C}(C_1, \ldots, C_n)$ and $C_k = \text{CES}_{\text{C}_\text{k}}(C_k^h, C_k^m), \;\; \forall k \in \{1, 2, \ldots, n\}$, etc., where CES_X is the according CES aggregator for some variable X. There is a simple iso-elastic downward-sloping export demand curve for every industry, i.e. $E_k = E_{0,k}(P_k^h)^{-\vartheta}$.[22] The total trade balance is simply the sum of the trade balances in each sector $TB = \sum_k TB_k$,

$$TB_k = P_k^h E_k - P_k^m(C_k^m + I_k^m + C_k^{G,m} + I_k^{G,m}) - \sum_j \left(P_j^m M_{jk}^m\right). \quad \text{(A.4)}$$

The value of total output is $\sum_k P_k^h F_k$, gross value added is $VA = \sum_k P_k \hat{Y}_k$ and both are related as follows:

$$\underbrace{\sum_k P_k^h F_k}_{\text{value of total output}} = \underbrace{\sum_k P_k \hat{Y}_k}_{\text{value added}} + \underbrace{\sum_{j,k} P_{jk}^{M,h} M_{jk}^h}_{\text{value of dom. intermediates}} + \underbrace{\sum_{j,k} P_{jk}^{M,m} M_{jk}^m}_{\text{value of imp. intermediates}}.$$

The following alterations to the system of excess demands have to be made,

[22] Note that export demand is now decreasing in P^h compared to P in (10.5).

where the single excess demand for final goods ζ^Y is replaced by ζ_k^F:

$$\zeta_k^F = D_k^h + \sum_k a_{jk}^h F_k - F_k, \; \forall k \tag{A.5}$$

$$\zeta^L = \sum_k L_k^D - L^S, \tag{A.6}$$

where total final demand for good k is given as $D_k^h = C_k^h + I_k^h + C_k^{G,h} + I_k^{G,h} + E_k$.[23] The last change is related to taxation. First, intermediates are subject to product taxes $T^M = \sum_{jk} \tau_{jk}^{M,h} P_i^h M_{jk}^h + \tau_{jk}^{M,m} P_i^m M_{jk}^m$ which enter government revenues and matter for calculation of $GDP = VA + T^C + T^M$. Second, lump-sum taxes and subsidies are measured in terms of the average production price $P^h = \sum_k P_k^h F_k / \sum_k F_k$.[24]

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[23] Note that the terminology 'sector-specific investment demand' is somewhat sloppy as one has to cleanly separate investment goods produced in sector k from investment goods used in sector k.

[24] P_t^h/P_0^h measures the GDP deflator (except for the price changes triggered by changes in product tax rates). Alternatively, lump-sum taxes and subsidies could easily be measured in terms of consumer prices P^C.

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